The Capon-MVDR Algorithm Threshold Region Performance Prediction and Its Probability of Resolution

Christ D. Richmond
MIT Lincoln Laboratory
244 Wood Street
Lexington, MA 02420-9108
phone: 781-981-5954
email: christ@ll.mit.edu

Abstract   The Capon-MVDR algorithm exhibits a threshold effect in mean-squared error (MSE) performance [1]. Below a specific threshold signal-to-noise ratio (SNR), the MSE of signal parameter estimates derived from the Capon algorithm rises rapidly. Prediction of this threshold SNR point is clearly of practical significance for system design and performance. Via an adaptation of an interval error-based method, referred to herein as the method of interval errors (MIE) [2],[3], the Capon threshold region MSE performance is accurately predicted. The exact pairwise error probabilities for the Capon (and Bartlett) algorithm, derived herein, are given by simple finite sums involving no numerical integration and include finite sample effects for an arbitrary colored data covariance. Combining these probabilities with the large sample MSE predictions of Vaidyanathan and Buckley [4], MIE provides accurate prediction of the threshold SNRs for an arbitrary number of well-separated sources, circumventing the need for numerous Monte Carlo simulations. A new two-point measure of the Capon probability of resolution is a serendipitous by-product of this analysis that predicts the SNRs required for closely spaced sources to be mutually resolvable by the Capon algorithm. These results represent very valuable design and analysis tools for any system employing the Capon-MVDR algorithm. Potential to characterize performance in the presence of mismatch is briefly considered.

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[2] F. Athley, Space Time Parameter Estimation in Radar Array Processing, PhD Dissertation, Chalmers University of Technology, Gothenburg, Sweden, June 2003.

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THE CAPON-MVDR ALGORITHM THRESHOLD REGION PERFORMANCE PREDICTION AND ITS PROBABILITY OF RESOLUTION

Christ D. Richmond

Lincoln Laboratory
Massachusetts Institute of Technology
244 Wood Street, Rm S4-317
Lexington, MA 02420
email: christ@ll.mit.edu

ABSTRACT

The threshold region mean squared error (MSE) performance of the Capon-MVDR algorithm is predicted via an adaptation of an interval error based method referred to herein as the method of interval errors (MIE). MIE requires good approximations of two quantities: (i) interval error probabilities, and (ii) the algorithm asymptotic (SNR→ ∞) MSE performance. Exact pairwise error probabilities for the Capon (and Bartlett) algorithm are derived herein that include finite sample effects for an arbitrary colored data covariance; with the Union Bound, accurate approximations of the interval error probabilities are obtained. Further, with the large sample MSE predictions of Vaidyanathan and Buckley, MIE accurately predicts the signal-to-noise ratio (SNR) threshold point, below which the Capon algorithm MSE performance degrades swiftly. A new exact two-point measure of the probability of resolution is defined for the Capon algorithm that accurately predicts the SNR at which sources of arbitrary closeness become resolvable.

1. INTRODUCTION

The threshold region mean squared error (MSE) performance of signal parameter estimates derived from the Capon high-resolution spectral estimator, *a.k.a* the minimum variance distortionless response (MVDR) spectral estimator, is the primary subject of this analysis. Similar to maximum-likelihood (ML) methods, the Capon processor is a beamscan type algorithm involving a nonlinear maximization of an objective search function (OSF). Parameter estimation algorithms requiring nonlinear searches typically exhibit a threshold effect in MSE performance. Below a specific signal-to-noise ratio (SNR) called the estimation threshold, the MSE departs from the asymptotic MSE performance and rises rapidly (see pp. 278–286 of [14]). Clearly, accurate prediction of this threshold SNR is of great practical significance for system design/analysis, particularly for methods capable of significant resolving power at SNRs too low for signal detection. Below the estimation threshold SNR, the MSE rises until it reaches a maximum that at times can be well approximated by the variance of an estimate that is assumed uniformly distributed over the search domain. The SNR

at which the MSE performance achieves this level of futility is called the no information point. Figure 1 illustrates this composite MSE performance typical of nonlinear estimation schemes. This composite MSE behavior is typical of nonlinear ML estimation [14], but likewise occurs with the Capon spectral estimator [15]. Although well-known, accurate prediction of this composite performance curve for the Capon algorithm remains an open problem. The goal of this analysis is to predict this MSE curve for the Capon algorithm with primary emphasis on threshold region performance from which an accurate prediction of the threshold SNR can be obtained.

A classical method of MSE approximation, referred to herein as the method of interval errors (MIE), was introduced by Van Trees [14] and provides a means of predicting threshold region performance of nonlinear estimation techniques. Variants of MIE have been applied to subspace based methods and ML estimation techniques with much success [10, 12, 16, 1, 8]. MIE requires good approximations of two quantities: (i) interval error probabilities, and (ii) the asymptotic MSE performance. Both of these quantities are algorithm dependent. The interval error probabilities quantify the likelihood that the estimator derives its signal parameter estimate from a false peak of the ambiguity function as opposed to the true peak. These probabilities are approximated via the Union Bound in conjunction with exact pairwise error probabilities for the Capon estimator that are derived herein; these derived probabilities account for arbitrary colored data covariance structure as well as finite sample support training effects [3, 7]. These calculations naturally lead to a two-point measure of the Capon probability of resolution from which accurate prediction of the SNRs required to resolve closely spaced sources is possible.

2. THE CAPON METHOD

The Capon high-resolution algorithm is well-known [2, 3, 15] and its performance has been studied extensively. It will be assumed in this section that all sources are well separated (by at least a beamwidth or Rayleigh distance) and possess SNRs that exceed the estimation threshold. In addition it is assumed that signals are mutually incoherent (coherent sources are not resolvable with the Capon algorithm), and that the total number of signals present in the data is known.

2.1. Capon's Approach

Given a set of independent identically distributed signal bearing observations $\mathbf{X} = [\mathbf{x}(1)|\mathbf{x}(2)|\cdots|\mathbf{x}(L)]$ where each vector is $N \times 1$ complex circular Gaussian, *i.e.* $\mathbf{x}(l) \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R})$, $l = 1, 2, \ldots, L$, Capon proposed the following power spectral estimator:

$$P_{Capon}(\theta) = \frac{1}{L - N + 1} \cdot \frac{1}{\mathbf{v}^H(\theta)\widehat{\mathbf{R}}^{-1}\mathbf{v}(\theta)} \qquad (1)$$

where $\mathbf{v}(\theta)$ is the assumed array response, $\widehat{\mathbf{R}} = \mathbf{X}\mathbf{X}^H$, and $\mathbf{R} = \mathbf{R}_N + \sigma_S^2 \cdot \mathbf{v}(\theta_T)\mathbf{v}^H(\theta_T)$, where $\mathbf{R}_N$ is background noise (possibly colored, but absent of signal-like interference). The maximum output provides an estimate of the signal power σ_S^2 and the signal parameter estimate is given by the scan value of θ that achieves this maximum; namely,

$$\widehat{\theta} = \arg \max_\theta P_{Capon}(\theta) \qquad (2)$$

(assuming a single signal is present). It shall be assumed that K signals are present in the data, and that the Capon parameter estimates $\widehat{\theta}_k$, $k = 1, 2, \ldots, K$, are obtained as the arguments of the K largest peaks of $P_{Capon}(\theta)$.

2.2. Large Sample MSE of the Capon Algorithm

The large sample ($L \gg N$) local error MSE performance of the Capon signal parameter estimator has been theoretically analyzed by several authors. Stoica et. al. [11], Vaidyanathan and Buckley (VB) [13], and Hawkes and Nehorai [5] exploit Taylor's theorem and complex gradient methods to approximate the MSE. VB provide an additional bias term via a second order Taylor series expansion that is particularly useful for capturing finite sample effects and a broader range of values for L and SNR. The results of VB will be used herein, and the local error MSE approximation obtained thereby shall be denoted by the symbol $\sigma_{VB}^2(\theta_k)$.

3. THRESHOLD REGION MSE PREDICTION

This section describes the method of interval errors (MIE) for MSE prediction and its adaptation to the Capon algorithm. The reader is also referred to [1, 16] for an excellent description of MIE in the context of ML estimation.

3.1. Method of Interval Errors

MIE builds upon the two regions of the composite MSE curve of Figure 1 that are given by the asymptotes of the SNR; namely, the no information (SNR$\to$ 0) and asymptotic (SNR$\to \infty$) regions. Define the conditioning event

$$\mathcal{A} = \{\text{True source parameters are } \theta_k, k = 1, 2, \ldots, K\}. \qquad (3)$$

MIE decomposes the MSE expression into two components: "no interval errors" (NIE), and "interval errors" (IE)

$$E\left\{ \left(\widehat{\theta}_k - \theta_k\right)^2 \Big| \mathcal{A} \right\} = \int p_{\widehat{\theta}_k}\left(\widehat{\theta}_k = \theta_0 \Big| \mathcal{A}\right)(\theta_0 - \theta_k)^2 \, d\theta_0$$
$$= \Pr\left(\text{NIE} \mid \mathcal{A}\right) E\left\{ \left(\widehat{\theta}_k - \theta_k\right)^2 \Big| \text{NIE}, \mathcal{A} \right\} +$$
$$\Pr\left(\text{IE} \mid \mathcal{A}\right) E\left\{ \left(\widehat{\theta}_k - \theta_k\right)^2 \Big| \text{IE}, \mathcal{A} \right\}$$

(see equation (127) on p. 282 of [14]). The parameter search space, *i.e.* the scanning domain for θ, is divided into disjoint mutually exclusive intervals based on the characteristics of underlying ambiguity function $\psi_{Capon}(\theta) \triangleq \frac{1}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$, that depends on $\mathbf{R}$, and hence is a function of the K SNRs of the K signals present.

3.1.1. Multiple Sources: $K \geq 1$

Assume arbitrary $K \geq 1$; in addition assume that these K signals are well separated by at least a beamwidth (thus, negligible likelihood of intersource errors). The extension of MIE to multiple sources is accomplished by expanding the "no interval errors" set to include all local neighborhoods of the K peaks in the ambiguity function due to the K sources present (clearly, all other intervals lead to IE). *The large sample MSE approximation obtained via $\sigma_{VB}^2(\theta_k)$ will be used to describe the "no interval errors" component contribution to the over MSE of the k-th source parameter Capon estimate.*

Let all local maxima within the signal parameter domain of interest of the ambiguity function when evaluated at K large SNRs (large enough that the ambiguity function has a local maximum at every true parameter value θ_k) be given by the finite set $\mathcal{M} = \{\theta \mid \theta_1, \theta_2, \ldots, \theta_{K+M-1}\}$ where θ_k for $k = 1, 2, \ldots, K$ represent the peaks due to the K sources, and θ_k for $k = K + 1, K + 2, \ldots, K + M - 1$ represent all other non-source local maxima.[1] The total MSE for this Capon parameter estimate can be approximated by

$$E\left\{ \left(\widehat{\theta}_k - \theta_k\right)^2 \Big| \mathcal{A} \right\} \simeq$$
$$\left[1 - \sum_{m=K+1}^{K+M-1} p\left(\widehat{\theta}_k = \theta_m \Big| \mathcal{A}\right)\right] \cdot \sigma_{VB}^2(\theta_k) \qquad (4)$$
$$+ \sum_{m=K+1}^{K+M-1} p\left(\widehat{\theta}_k = \theta_m \Big| \mathcal{A}\right)(\theta_m - \theta_k)^2.$$

The interval error probability $p\left(\widehat{\theta}_k = \theta_m \Big| \mathcal{A}\right)$ represents the likelihood of the Capon search algorithm choosing a value associated with the false peak located at $\theta = \theta_m$ as an estimate for θ_k, when the K true signals are located at parameter values $\theta = \theta_k$, $k = 1, 2, \ldots, K$.

As in [1, 16], the dominant term of the Union Bound (UB) can be used to approximate the interval error probabilities:

$$p\left(\widehat{\theta}_k = \theta_m \Big| \mathcal{A}\right) \simeq$$
$$\Pr\left[P_{Capon}(\theta_m) > P_{Capon}(\theta_k)| \mathcal{A}\right]. \qquad (5)$$

This modified UB approximation is remarkably accurate in the vicinity of the estimation threshold SNR, but tends to over predict the MSE in the no information region. *Thus, the minimum of (4) and the worse case MSE obtained with an estimate $\widehat{\theta}_k$ that is uniformly distributed over the parameter search space will be chosen as the MSE prediction.*

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[1] Such SNRs will exist provided that no array response mismatch is present, *i.e.* provided that the array responses used to compute $P_{Capon}(\theta)$ match the K array responses existing in the true data covariance $\mathbf{R}$ for θ_k, $k = 1, 2, \ldots, K$.

3.2. Capon Pairwise Error Probabilities

The desired pairwise error probabilities are of the form

$$P_e^{Capon}(\theta_a|\theta_b) \triangleq \Pr\left[P_{Capon}(\theta_a) > P_{Capon}(\theta_b)\,|\,\mathcal{A}\right]. \qquad (6)$$

Define the following function

$$\mathcal{F}(x, N_0) \triangleq \frac{x^{N_0}}{(1+x)^{2N_0-1}} \sum_{k=0}^{N_0-1} \binom{2N_0-1}{k+N_0} \cdot x^k \qquad (7)$$

where $\mathcal{F}(x, N_0)$ is the cumulative distribution function for a special case of the complex central F statistic. The algorithm for computing the pairwise error probabilities for the Capon estimator is as follows:

1. Define the $N \times 2$ matrix $\mathbf{V} = [\mathbf{v}(\theta_a)|\mathbf{v}(\theta_b)]$ and choose the desired covariance parameter $\mathbf{R}$.

2. Perform the following QR-decomposition

$$\mathbf{R}^{-1/2}\mathbf{V} = \mathbf{Q}^H \left[\begin{array}{c} \boldsymbol{\Delta}_{2\times2} \\ \mathbf{0}_{(N-2)\times2} \end{array}\right]; \text{let } \boldsymbol{\Delta} = [\boldsymbol{\delta}_1|\boldsymbol{\delta}_2]. \quad (8)$$

3. Define the matrix $\boldsymbol{\delta}_2\boldsymbol{\delta}_2^H + F \cdot \boldsymbol{\delta}_1\boldsymbol{\delta}_1^H$ for any non-positive real number $F \le 0$, and its two eigenvalues as $\lambda_1(F)$ and $\lambda_2(F)$, and their ratio as $l_\lambda(F) = -\lambda_2(F)/\lambda_1(F)$.

4. The desired exact pairwise error probability for the Capon algorithm is given by the expression

$$\boxed{\begin{array}{c} P_e^{Capon}(\theta_a|\theta_b) = 0.5 \cdot \{1 + \text{sign}\,[\lambda_1(-1)]\} \\ -\text{sign}[\lambda_1(-1)] \cdot \mathcal{F}[l_\lambda(-1), L - N + 2]. \end{array}} \qquad (9)$$

See [9] for derivation.

4. THE CAPON PROBABILITY OF RESOLUTION

A useful measure of the probability of resolution can be defined that provides excellent prediction of the SNR at which sources can be resolved by the Capon algorithm. For a two closely spaced sources scenario of the form $\mathbf{R} = \mathbf{R}_N + \sigma_{S_a}^2 \mathbf{v}(\theta_0)\mathbf{v}^H(\theta_0) + \sigma_{S_b}^2 \mathbf{v}(\theta_0 + \delta\theta)\mathbf{v}^H(\theta_0 + \delta\theta)$, define parameter θ_{MP} as the parameter value of the source with the smallest power out of the ambiguity function, *i.e.* $\theta_{MP} \triangleq \arg \min_{\theta_0, \theta_0+\delta\theta} \psi_{Capon}(\theta)$. A two point measure of the probability of resolution can be defined as

$$\begin{array}{c} P_{res}^{Capon}(\theta_0, \theta_0 + \delta\theta) \triangleq \\ \Pr\left[P_{Capon}\left(\theta_0 + \frac{\delta\theta}{2}\right) \le \rho \cdot P_{Capon}(\theta_{MP})\right] \end{array} \qquad (10)$$

where $0 \le \rho \le 1$. The parameter ρ essentially defines the desired "dip" in Capon output power between two closely spaced sources. For example, if $\sigma_{S_a}^2 = \sigma_{S_b}^2$ and $\rho = 0.5$, then $P_{res}^{Capon}(\theta_0, \theta_0+\delta\theta)$ is the probability that the dip in $P_{Capon}(\theta)$ midway between these two sources is at least 3dB less than $P_{Capon}(\theta)$ evaluated at either source location. Similar measures of resolution have been proposed [4, 15]. The algorithm for computing the Capon two point probability of resolution is the same as that for $P_e^{Capon}(\theta_a|\theta_b)$ with $\theta_a = \theta_{MP}$, $\theta_b = \theta_0 + \delta\theta/2$, and $F = -1/\rho$; namely, the

desired two point measure of the probability of resolution is given by

$$\boxed{\begin{array}{c} P_{res}^{Capon}(\theta_0, \theta_0 + \delta\theta) = 0.5 \cdot \{1 + \text{sign}\,[\lambda_1(-1/\rho)]\} \\ -\text{sign}[\lambda_1(-1/\rho)] \cdot \mathcal{F}[l_\lambda(-1/\rho), L - N + 2]. \end{array}} \qquad (11)$$

A detailed discussion of performance with closely spaced sources utilizing this measure is given in [9].

5. NUMERICAL EXAMPLES

5.1. Ex. 1: Single Broadside Signal in White Noise

Consider a Direction of Arrival (DOA) estimation scenario involving a single source and a set of signal bearing snapshots $\mathbf{x}(l) \sim \mathcal{CN}[\mathbf{0}, \mathbf{I} + \sigma_S^2\mathbf{v}(\theta_T)\mathbf{v}^H(\theta_T)]$, $l = 1, 2, \ldots, L$, for an $N = 18$ element uniform linear array (ULA) with slightly less than $\lambda/2$ element spacing. The array has a 3dB beamwidth of 7.2 degrees and the desired target signal is arbitrarily placed at $\theta_T = 90$ degrees (array broadside). The signal parameter search space of interest is defined to be $\theta \in [60°, 120°]$. The signal parameter to be estimated is simply the scalar angle of arrival $\theta = \theta_T$. Figure 2 illustrates the Monte Carlo based MSE performance of the Capon algorithm alongside its MIE prediction and the Cramér-Rao Bound (CRB) for sample support cases $L = 1.5N, 2N$ and $3N$ snapshots, all plotted as a function of the element level SNR. The Capon estimator clearly is not asymptotically (L fixed, SNR$\rightarrow \infty$) efficient, since increasing the SNR does not bring its MSE performance closer to the CRB, hence the need for analyses such as [11, 13, 5]. The VB MSE prediction is plotted for the $L = 2N$ case to illustrate that this large sample Taylor Series based approximation is a local one and is only valid above the estimation threshold SNR. The goal of MIE is to reasonably predict MSE performance well into the estimation threshold region. Note from the $L = 2N$ case in Figure 2 that MIE continues with accurate prediction well into the threshold region by accounting for global errors, whereas the VB MSE prediction becomes inaccurate. For example, the VB prediction is off by about 5dB for the SNR required for a 6 to 1 beam split ratio (RMSE$\simeq -7.5$dB). Note that in the no information region the UB approximation begins to over predict the MSE. Thus, it is allowed to increase until it maxes at the MSE obtained for an estimate that is uniformly distributed over the signal parameter domain of interest.

5.2. Ex. 2: Multiple Signals in White Noise

Next consider the same scenario, but with an additional source of equal power included in the environment at 70 degrees. The MSE performance of both signal parameter estimates is illustrated in Figures 3–4. The MIE predictions remain quite accurate.

5.3. Ex. 3: Tilted Minimum Redundancy Linear Array

MIE can be modified to account for the presence of signal model mismatch. The details of this modification are discussed in [9] and have been applied extensively with much success to the adaptive matched field source localization problem in [6]. As an illustration, consider an $N = 4$ element minimum redundancy linear array (MRLA) [15] that has much higher sidelobes (ambiguities)

than a fully populated ULA (see beampatterns in Figure 5). Mismatch can be introduced by tilting this array, yet processing the data as if the array were not tilted. MIE can account for this mismatch as illustrated in Figure 6, where several tilt angles have been considered, and MSE is plotted as a function of the output array SNR. Note that the MSE predictions accurately capture both the global errors due to the high ambiguities (threshold region), as well as the asymptotic bias resulting from mismatch. Such encompassing predictions have never been made for the Capon algorithm.

5.4. Ex. 4: Probability of Resolution

The two point measure of the probability of resolution proposed in Section 4 allows one to predict the SNRs required for a dip to appear between closely spaced sources with high confidence. This measure of resolution is computable for any chosen pair of scanning vectors $\mathbf{v}(\theta_a), \mathbf{v}(\theta_b)$, and any choice of data covariance structure $\mathbf{R}$. Thus, deterministic and stochastic mismatch can be easily introduced into analysis.

As a last example, consider the $N = 18$ element ULA of example 2 of Section 5.2. Let the additional source of equal power be placed at 93 degrees, *i.e.* at less than half a beamwidth separation. Let the nominal ULA array sensor positions $\mathbf{z}_n$ be independently perturbed by zero mean spherically symmetric Gaussian noise, such that the true array sensor positions are given by $\mathbf{z}_n + \mathbf{e}_n$ where $\mathbf{e}_n \sim \mathcal{N}_3(\mathbf{0}, \mathbf{I}_3 \sigma_{RMS}^2)$. Figure 7 shows a plot of the probability of resolution as a function of the output array SNR with $L = 2N$ spatial snapshots. Two forms of mismatch are considered: (i) deterministic, and (ii) stochastic. The leftmost plot was generated by simply using a single realization of a perturbed array as the root MSE of the perturbations is steadily increased (different perturbations used for each value of σ_{RMS}^2). The rightmost plot represent the resolution observed as one averages over an ensemble of perturbation realizations. Note that the knee in the curve appears to be relatively constant as a function of SNR; namely, the maximal resolution is achieved at an array SNR of approximately 40dB. SNR in excess of this value is essentially wasteful. As more mismatch is introduced the asymptotic likelihood of resolution decreases.

Lastly, note that the dip in P_{res}^{Capon} before its eventual rise is simply due to the fact that the underlying Capon ambiguity function has a single peak midway between the two sources for low SNRs values (sources are unresolved). It is only when the source SNRs exceed that necessary for resolution that two distinct peaks appear at the source locations, with a dip in between. Since the two points used for the probability of resolution calculation are chosen such that one is at one of the source locations and the other midway between the two sources, a dip occurs before the rise.

6. CONCLUSIONS

The method of interval errors (MIE) has been successfully adapted and extended to the Capon-MVDR algorithm, providing remarkably accurate prediction of the MSE threshold SNRs for an arbitrary number of well separated sources. These SNRs are predicted via simple finite sum expressions for the pairwise error probabilities, involving no numerical integration, and circumventing the need for many time consuming and cumbersome Monte Carlo simulations. A new two-point measure of the Capon probability of resolution was proposed that accurately predicts the SNRs necessary for mutual source resolvability for sources of arbitrary close-

ness. Both account for colored noise, finite sample effects, and signal model mismatch, and thus, represent valuable design/analysis tools for any system employing the Capon algorithm.

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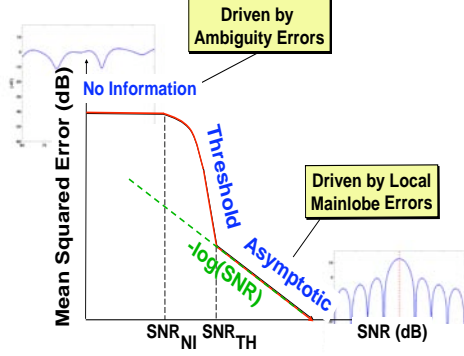
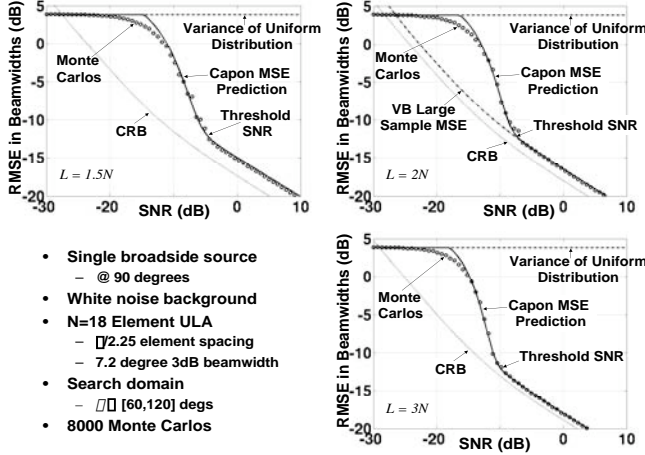**Fig. 1**. Composite MSE Curve for Parameter Estimation

- **Single broadside source**
  - @ 90 degrees
- **White noise background**
- **N=18 Element ULA**
  - λ/2.25 element spacing
  - 7.2 degree 3dB beamwidth
- **Search domain**
  - $\theta \in$ [60,120] degs
- **8000 Monte Carlos**

Fig. 2. Single Source Capon MSE Performance, $\theta_T = 90°$, $L = 1.5N, 2N, 3N$

[16] W. Xu, *Performance Bounds on Matched-Field Methods for Source Localization and Estimation of Ocean Environmental Parameters*, Ph.D. Dissertation, MIT, Cambridge, MA, June 2001.

- **Two sources**
  - @ 90 and 70 degrees
- **Results for 90 deg source**
- **White noise background**
- **N=18 Element ULA**
  - λ/2.25 element spacing
  - 7.2 degree 3dB beamwidth
- **Search domain**
  - $\theta \in$ [60,120] degs
- **1500 Monte Carlos**

Fig. 3. Two Source Capon MSE Performance, $\theta_1 = 90°$, $\theta_2 = 70°$, $L = 1.5N, 2N, 3N$

- **Two sources**
  - @ 90 and 70 degrees
- **Results for 70 deg source**
- **White noise background**
- **N=18 Element ULA**
  - λ/2.25 element spacing
  - 7.2 degree 3dB beamwidth
- **Search domain**
  - $\theta \in$ [60,120] degs
- **1500 Monte Carlos**
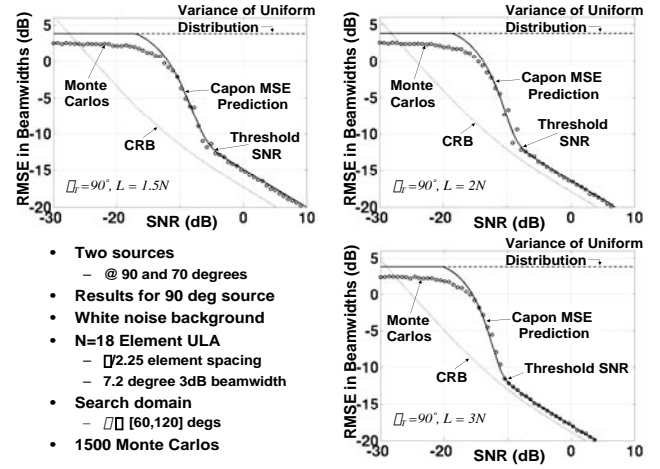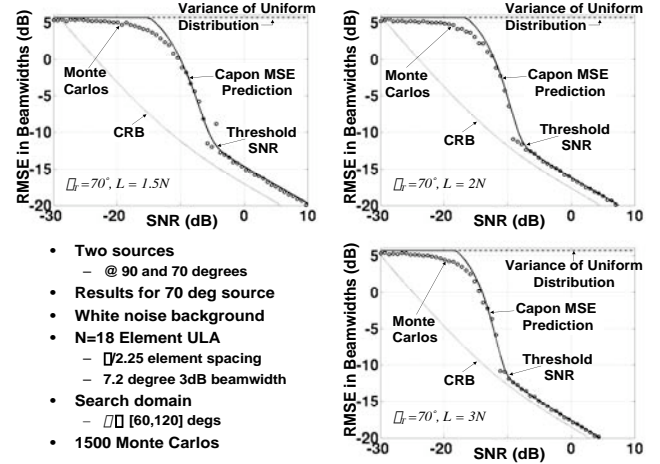
Fig. 4. Two Source Capon MSE Performance, $\theta_1 = 90°$, $\theta_2 = 70°$, $L = 1.5N, 2N, 3N$
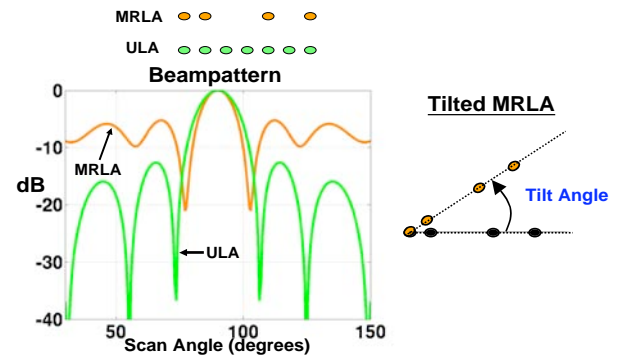
Fig. 5. Minimum Redundancy Linear Array: High ambiguities relative to fully populated ULA
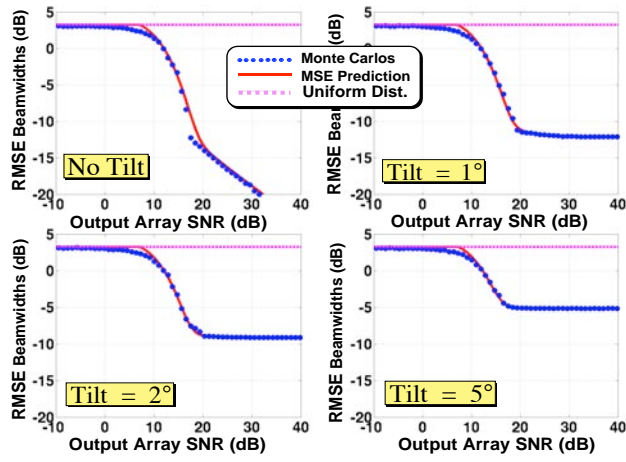
Fig. 6. MSE for Tilted MRLA with signal at broadside and $L = 3N$
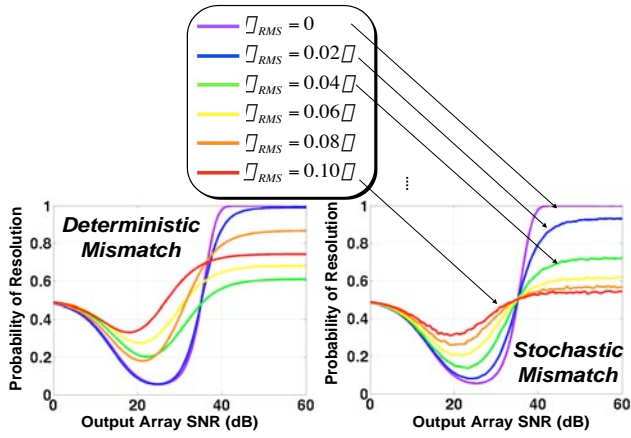
Fig. 7. Capon Probability of Resolution, $\theta_0 = 90°$, $\theta_0 + \delta\theta = 93°$, $L = 2N$

The Capon-MVDR Algorithm Threshold Region Performance Prediction and its Probability of Resolution

Christ D. Richmond

The Adaptive Sensor Array Processing Workshop
MIT Lincoln Laboratory

Session 2: Detection and Estimation
1:30pm, Tuesday, March 16th 2004

MIT Lincoln Laboratory

Outline

✓ • **Introduction**
   – **Historical Perspective**
   – **Goals of Analysis / Previous Work**

• Capon Algorithm

• Theory of MSE Prediction

• Numerical Examples

• Conclusions

Historical Perspective*:
Spectral Analysis and Super Resolution

Discrete

Continuous

Super-Resolution

Time Line

Pythagoras B.C.

Newton 1671, 1687

Bernoulli 1738

Euler 1755

Fourier 1822

Schuster 1897

Michelson -Stratton 1898

Einstein 1914

Yule 1927

Wiener 1930

Walker 1931

Blackman 1958

Tukey

Cooley 1965

Burg 1967

Capon 1969

Schmidt 1979

$$S_x(f)$$

f

*S. L. Marple, Jr. 1987, E. Robinson 1985

Historical Perspective*:
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Cooley 1965

Burg 1967

Capon 1969

Schmidt 1979

Time Line

$S_x(f)$

$f_0 \qquad 2f_0 \qquad 3f_0$

f

*S. L. Marple, Jr. 1987, E. Robinson 1985

Historical Perspective*:
Spectral Analysis and Super Resolution

Discrete   **Continuous**   **Super-Resolution**

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Walker 1931

Blackman 1958

Tukey

Cooley 1965

Burg 1967

Capon 1969

Schmidt 1979

$S_x(f)$

Area = Power in Band

f_0

f

C. D. Richmond-3
Tuesday,16th March 2004
ASAP 2004

*S. L. Marple, Jr. 1987, E. Robinson 1985

Historical Perspective*:
Spectral Analysis and Super Resolution

Discrete **Continuous** **Super-Resolution**

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Time Line

Techniques
Maximum Entropy
Capon Algorithm
MUSIC / SVD Based
ESPRIT
Maximum Likelihood
APES

$S_x(f)$

$$f_0 + \delta f \quad f_0$$

f

*S. L. Marple, Jr. 1987, E. Robinson 1985

MIT Lincoln Laboratory

Historical Perspective*:
Spectral Analysis and Super Resolution

Discrete

Continuous

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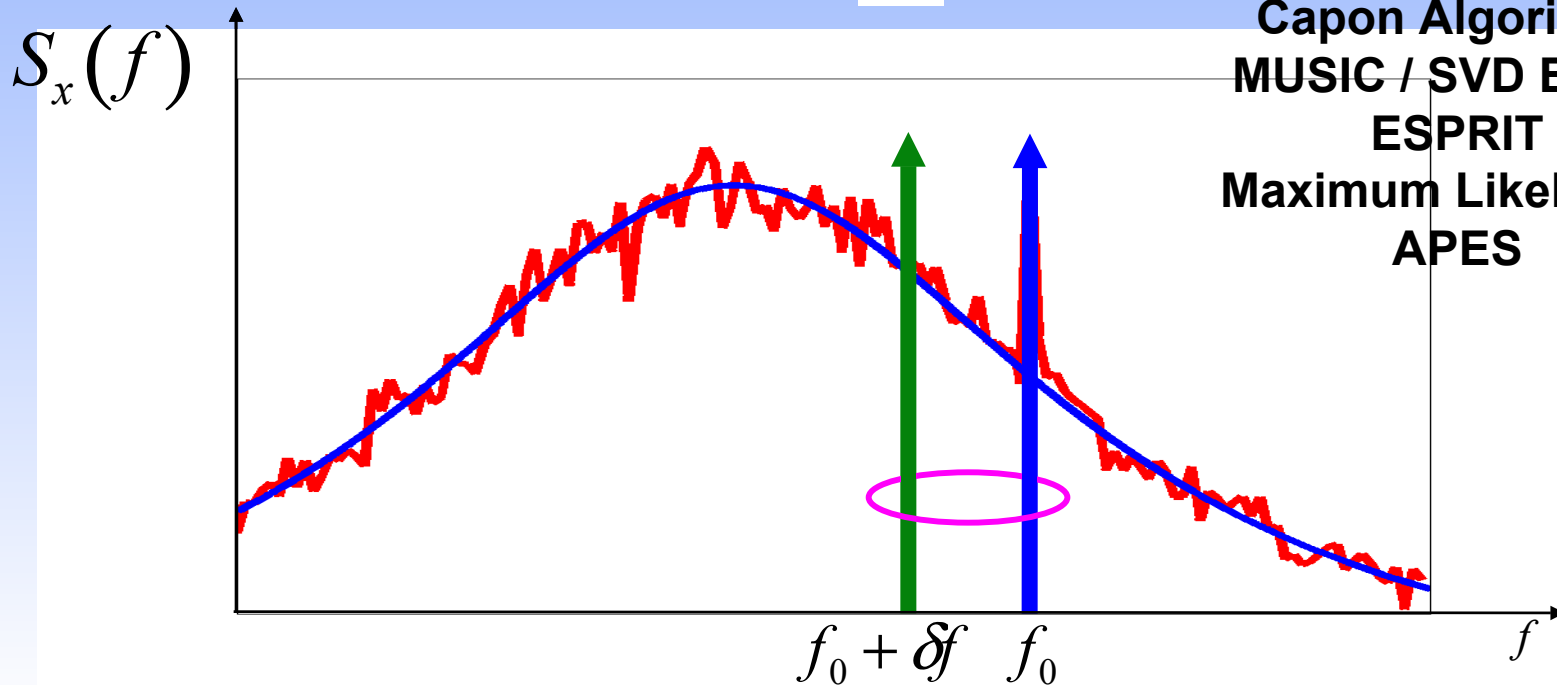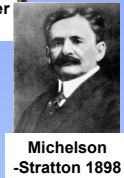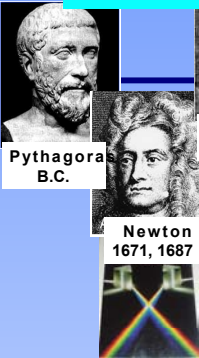
Robust **➡**
High-Resolution *Time Line*

Today

Techniques
Maximum Entropy
Capon Algorithm
MUSIC / SVD Based
ESPRIT
Maximum Likelihood
APES

$S_x(f)$

$f_0 + \delta f$ f_0

f

S. L. Marple, Jr. 1987, E. Robinson 1985

MIT Lincoln Laboratory

Goals of Analysis

- **Problem:**

    - **Mean Squared Error (MSE) performance of Capon signal parameter estimation unknown non-asymptotically**

        1. **Colored Noise (CLR)**

        2. **Signal Modeling Errors (MIS)**

        3. **Adaptive (unknown data covariance) Finite Training (FIN)**

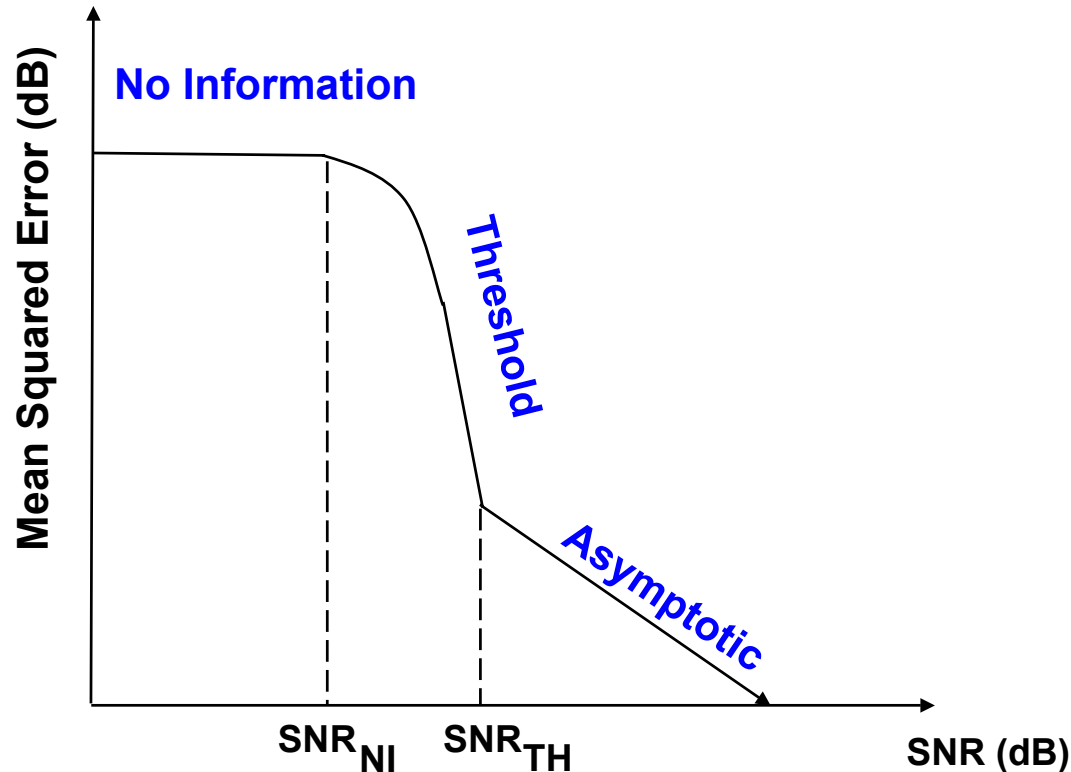   - **Capon Probability of Resolution (RES) unknown (1-3 above also)**

- **Goal:**

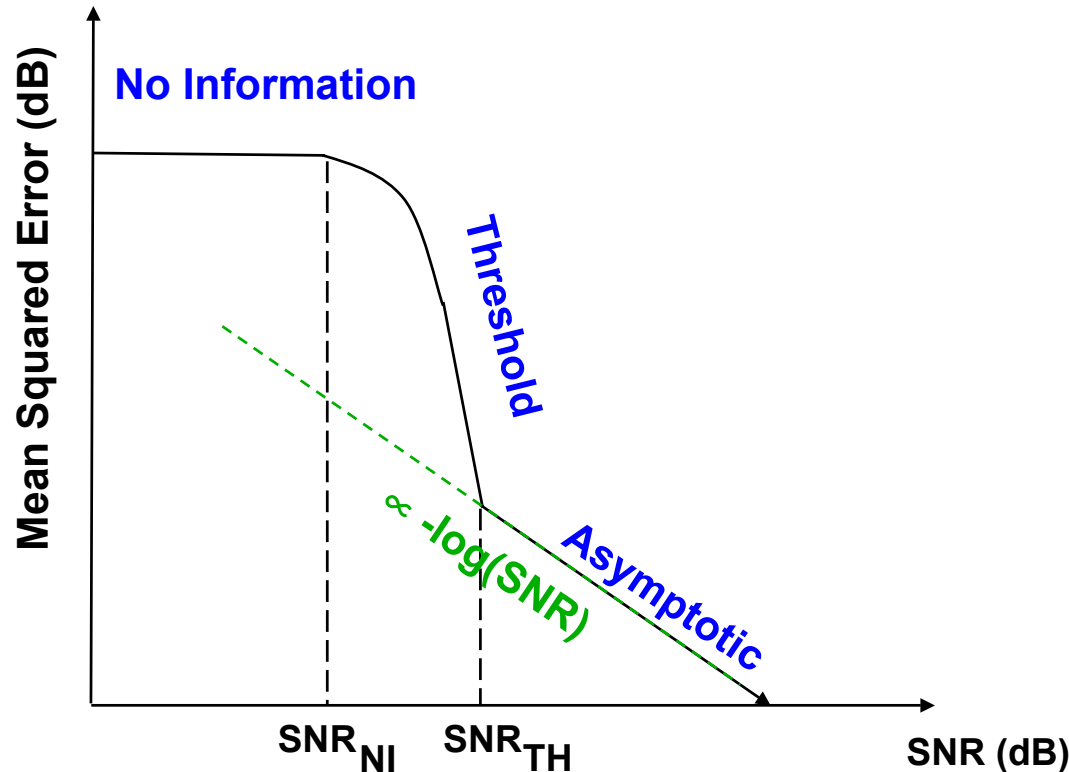   - **Develop robust theory for prediction of Capon non-asymptotic MSE and probability of resolution**

Goals of Analysis

Capon Spectral Estimator: **Mean Squared Error:**

$$\hat{\theta}_k = \arg\, k\text{th largest peak}_{\theta}\, \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)} \implies E\left\{\left(\hat{\theta}_k - \theta_k\right)^2 \middle| \mathbf{R}, \mathbf{v}(\theta), L, N\right\}$$

- **Problem:**

  - **Mean Squared Error (MSE) performance of Capon signal parameter estimation unknown non-asymptotically**

    1. **Colored Noise (CLR)**

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- **Goal:**

  - **Develop robust theory for prediction of Capon non-asymptotic MSE and probability of resolution**

Typical Composite MSE Performance: Nonlinear Estimation "101"

- **Three definitive regions of Signal-to-Noise-Ratio (SNR)**
  - No Information, Threshold/Ambiguity, and Asymptotic
- **Recall MSE = Estimator Variance + Estimator Bias**

Typical Composite MSE Performance: Nonlinear Estimation "101"

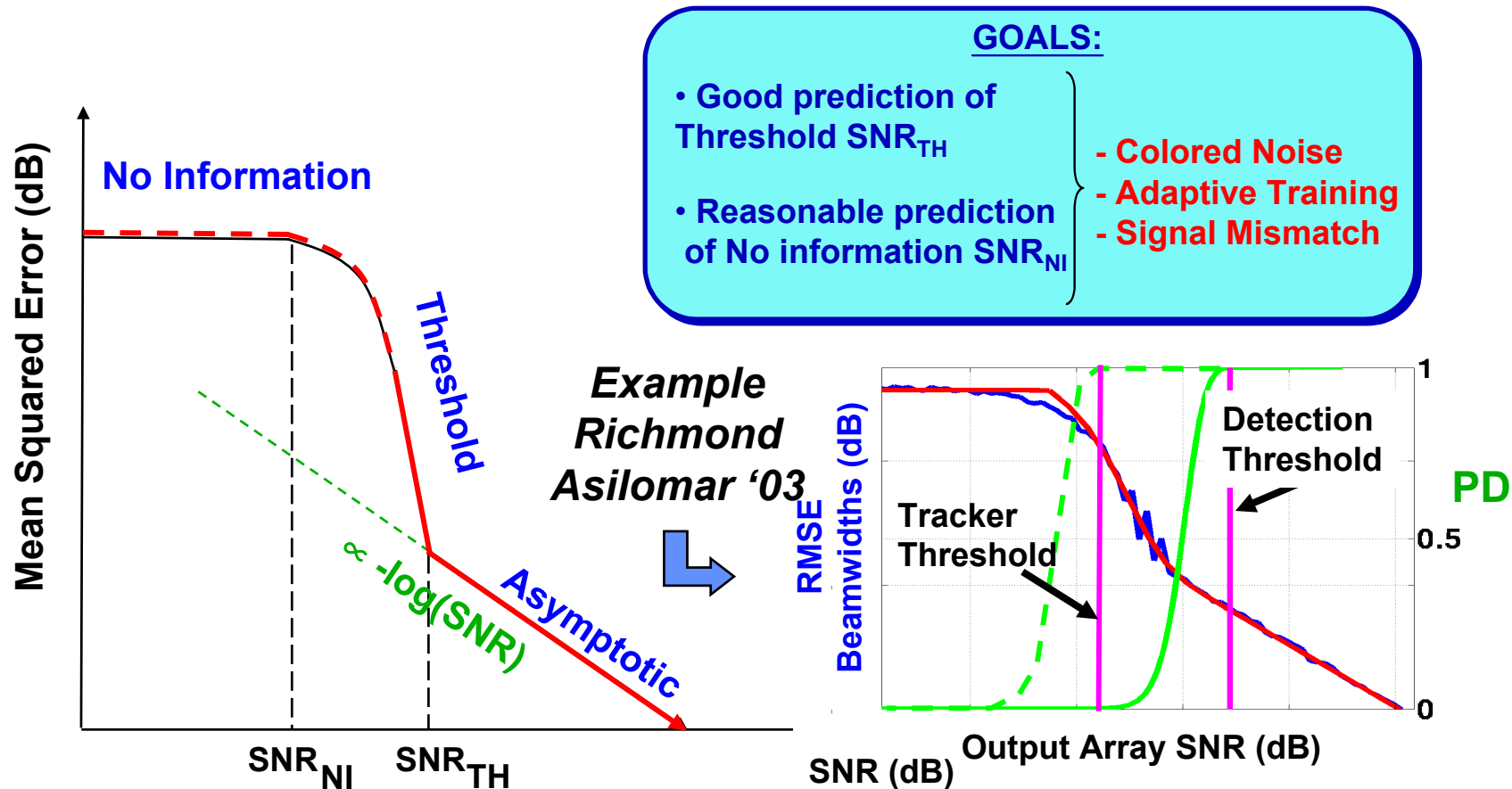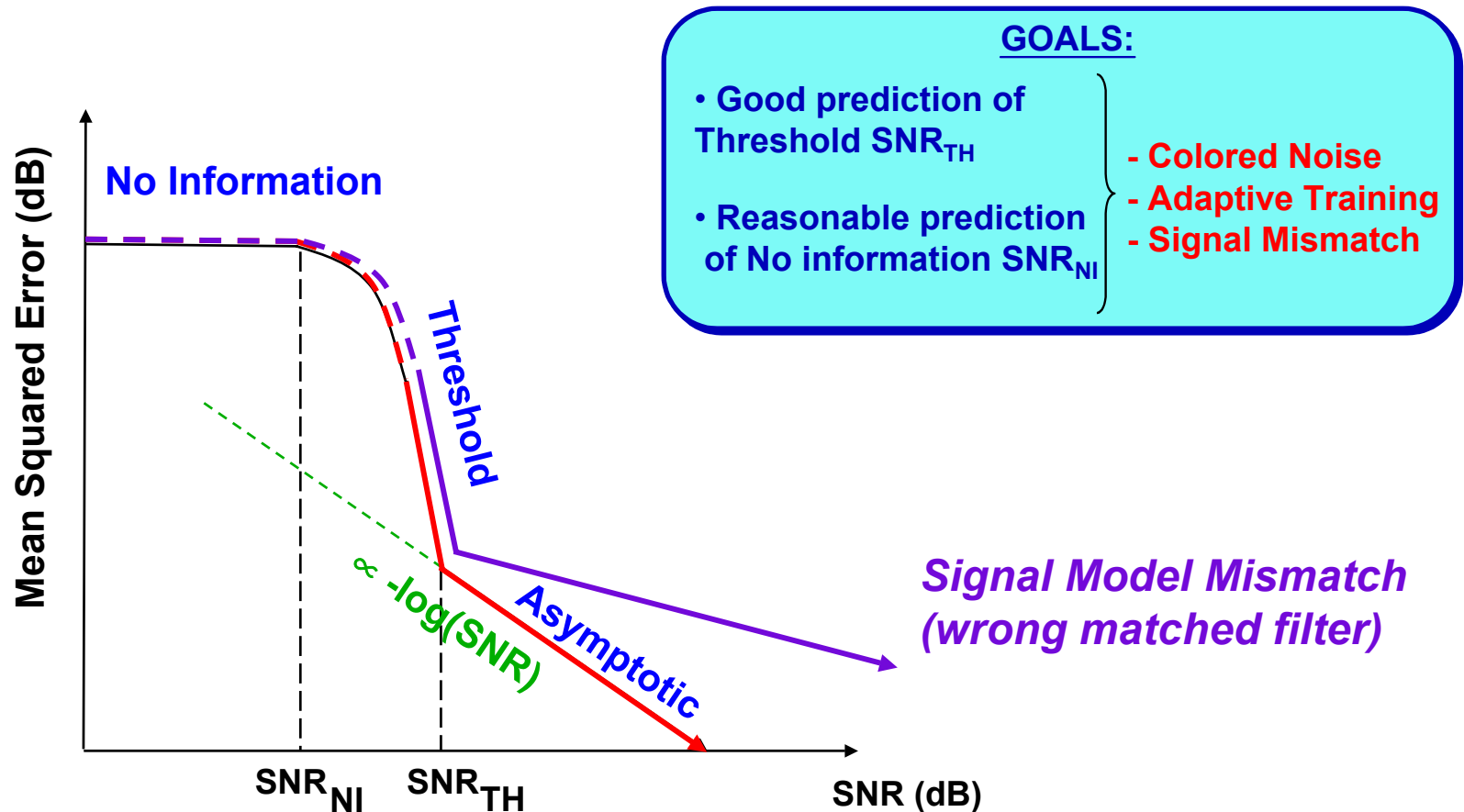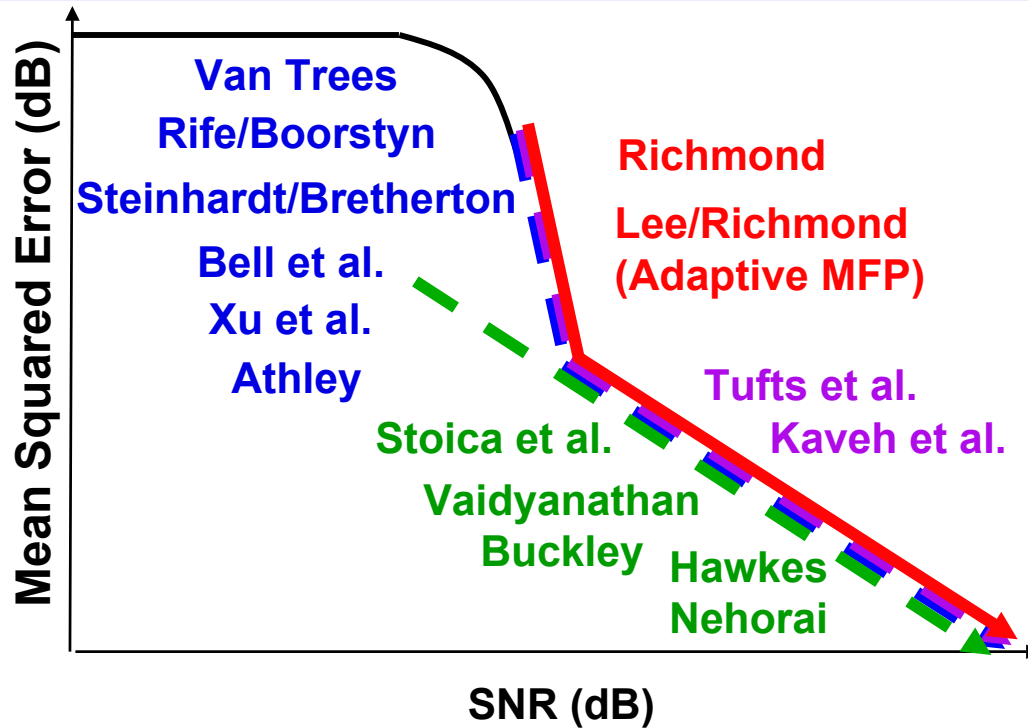- **Three definitive regions of Signal-to-Noise-Ratio (SNR)**
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Typical Composite MSE Performance: Nonlinear Estimation "101"

Driven by Global Ambiguity Errors

Ambiguity Function:
Output of matched filter as parameters of assumed signal hypothesis are varied

No Information

Mean Squared Error (dB)

Threshold

$\propto$ **-log(SNR)**

Asymptotic

Driven by Local Mainlobe Errors

SNR_{NI} SNR_{TH}

SNR (dB)

- **Three definitive regions of Signal-to-Noise-Ratio (SNR)**
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Typical Composite MSE Performance: Nonlinear Estimation "101"

GOALS:

- Good prediction of Threshold SNR_{TH}
- Reasonable prediction of No information SNR_{NI}

- Colored Noise
- Adaptive Training
- Signal Mismatch

No Information

Threshold

$\propto -\log(SNR)$

Asymptotic

Mean Squared Error (dB)

SNR_{NI} SNR_{TH}

SNR (dB)

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GOALS:

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- **Reasonable prediction of No information SNR_{NI}**

  - **Colored Noise**
  - **Adaptive Training**
  - **Signal Mismatch**

No Information

Mean Squared Error (dB)

Threshold

$\propto$ -log(SNR)

Asymptotic

SNR_{NI} SNR_{TH}

Example Richmond Asilomar '03

RMSE Beamwidths (dB)

Tracker Threshold

Detection Threshold

PD

1

0.5

0

SNR (dB)   **Output Array SNR (dB)**

- **Three definitive regions of Signal-to-Noise-Ratio (SNR)**
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Typical Composite MSE Performance: Nonlinear Estimation "101"

GOALS:

- Good prediction of Threshold SNR_{TH}
- Reasonable prediction of No information SNR_{NI}

- Colored Noise
- Adaptive Training
- Signal Mismatch

No Information

Threshold

Asymptotic

$\propto -\log(SNR)$

Signal Model Mismatch (wrong matched filter)

Mean Squared Error (dB)

SNR_{NI} SNR_{TH} SNR (dB)

- **Three definitive regions of Signal-to-Noise-Ratio (SNR)**
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- **Recall MSE = Estimator Variance + Estimator Bias**

Previous Work and New Contributions

Mean Squared Error (dB) vs SNR (dB)

Van Trees
Rife/Boorstyn
Steinhardt/Bretherton
Bell et al.
Xu et al.
Athley
Richmond
Lee/Richmond (Adaptive MFP)
Stoica et al.
Vaidyanathan
Buckley
Hawkes Nehorai
Tufts et al.
Kaveh et al.

— *Capon:* **Adaptive, Colored Noise (CLR), Finite Sample Effects (FIN), Mismatch (MIS), Prob. Resolution (RES)**

– – *Capon*: **Asymptotic Only, (Large SNR, Many Samples)**

– – *Subspace Methods:* **Large Sample**

– – *Maximum Likelihood* **(ML): Non-Adaptive, White Noise (WHT)**

Capon Algorithm MSE Analyses

ASYMPTOTIC REGION					THRESHOLD REGION				
WHT	CLR	MIS	FIN	RES	WHT	CLR	MIS	FIN	RES
				NEW	NEW	NEW	NEW	NEW	NEW

Outline

- **Introduction**
- ✓ **Capon Algorithm**
- **Theory of MSE Prediction**
- **Numerical Examples**
- **Conclusions**

Capon-MVDR Spectral Estimator: High-Resolution Algorithm

• **Capon proposed to _design linear filter optimally_:**

Let data covariance be $\mathbf{R} = E\left\{\mathbf{x}_l \mathbf{x}_l^H\right\}$ **for** $l = 1, 2, \mathrm{K}, L$

Choose filter weights w according to

$$\min \mathbf{w}^H \mathbf{R} \mathbf{w} \quad \text{such that} \quad \mathbf{w}^H \mathbf{v}(f_0) = 1$$

Minimum Variance   *Distortionless Response*

Solution well-known:

Average Output Power of Optimal Filter:

$$\mathbf{w}(f_0) = \frac{\mathbf{R}^{-1}\mathbf{v}(f_0)}{\mathbf{v}^H(f_0)\mathbf{R}^{-1}\mathbf{v}(f_0)} \quad \Longrightarrow \quad E\left\{\left|\mathbf{w}^H(f_0)\mathbf{x}_l\right|^2\right\} = \frac{1}{\mathbf{v}^H(f_0)\mathbf{R}^{-1}\mathbf{v}(f_0)}$$

Capon-MVDR Spectral Estimator: High-Resolution Algorithm

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- **Capon suggested the following practical implementation:**

  $$P_{Capon}(f_0) \equiv \frac{1}{\mathbf{v}^H(f_0)\hat{\mathbf{R}}^{-1}\mathbf{v}(f_0)} \qquad \text{where} \qquad \hat{\mathbf{R}} \equiv \frac{1}{L}\sum_{l=1}^{L}\mathbf{x}_l \mathbf{x}_l^H$$

  Can use to estimate frequencies of pure tones

  $$\hat{f}_0 = \arg\max_f \frac{1}{\mathbf{v}^H(f)\hat{\mathbf{R}}^{-1}\mathbf{v}(f)}$$

  $f_0 - \delta f \quad f_0$
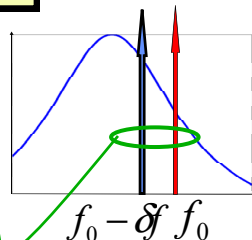
Super-resolution possible superior to FFT/CBF.

Outline

- **Introduction**
- **Capon Algorithm**
- ✓ **Theory of MSE Prediction**
    - **Nonlinear Estimation / Ambiguity Functions**
    - **Method of Interval Errors**
    - **Capon Pairwise Error Probabilities**
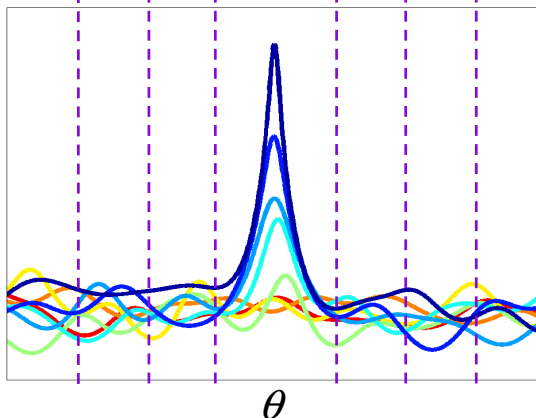- **Numerical Examples**
- **Conclusions**

Nonlinear Estimation and Ambiguity Functions

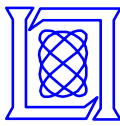$$\hat{\theta} = \arg\max_{\theta} \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$
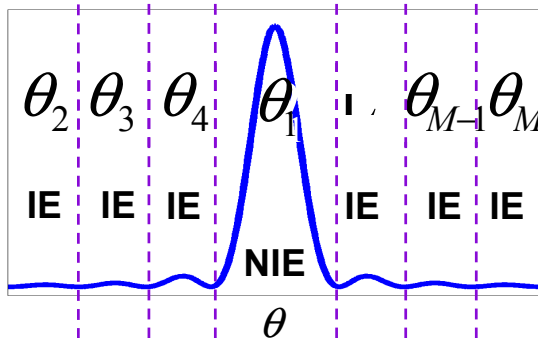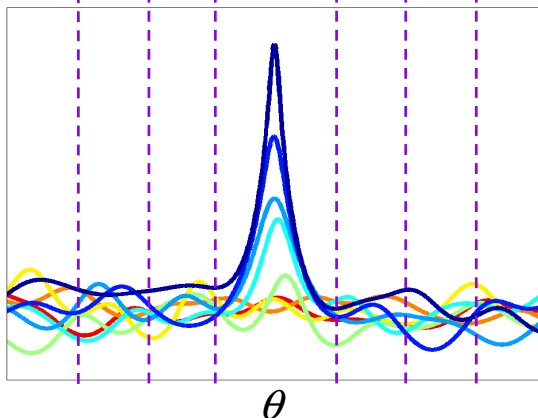
**Capon
Ambiguity Function**

$$\frac{1}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$$

θ

$$\frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

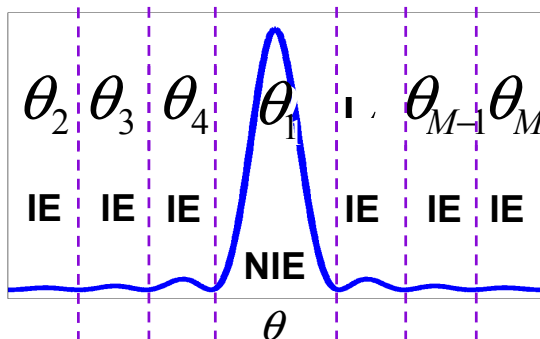**Stochastic Realizations**

Increasing SNR

θ

- **Capon algorithm involves nonlinear search on finite interval**

- **Estimate $\hat{\theta}$ is the position of the maximum of a non-stationary non-Gaussian stochastic process on a finite interval**
  - **Driven by multimodal Ambiguity Function**

- **Interval can be divided into M sub-intervals**
  - **"No Interval Error" (NIE)**
    - **Local Errors**
  - **"Intervals of Error" (IE)**
    - **Global Errors**

- **Use intervals to approximate MSE**
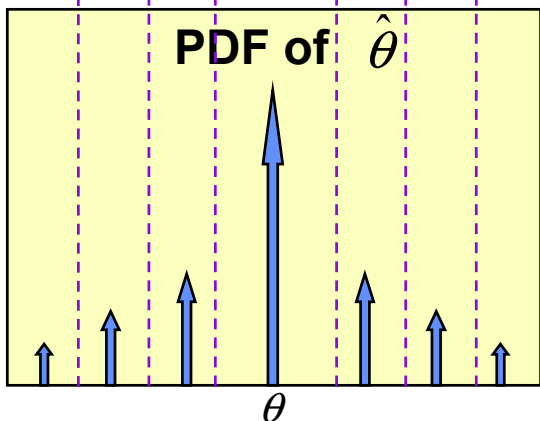 - **Estimation process approximated by M-ary hypothesis testing problem**

Nonlinear Estimation and Ambiguity Functions

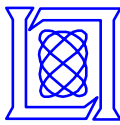$$\hat{\theta} = \arg\max_{\theta} \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

Capon Ambiguity Function

$$\frac{1}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$$

IE | IE | IE | NIE | IE | IE | IE

θ

$$\frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

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Increasing SNR

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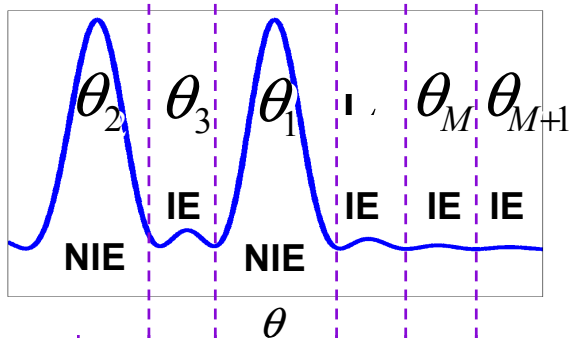 - **Estimation process approximated by M-ary hypothesis testing problem**

Nonlinear Estimation and Ambiguity Functions

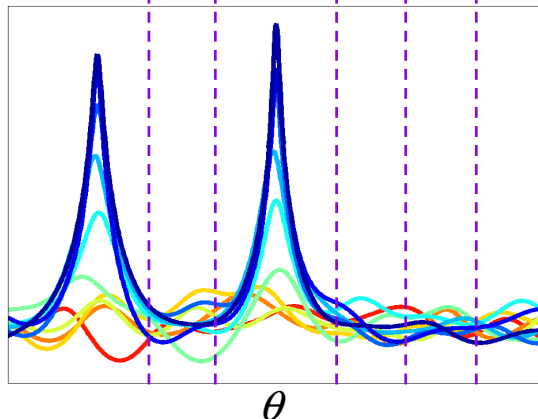$$\hat{\theta} = \arg\max_{\theta} \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

Capon Ambiguity Function

$$\frac{1}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$$

θ_2 θ_3 θ_4 θ_1 θ_{M-1} θ_M

IE IE IE   IE IE IE

NIE

θ

$$\frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

Stochastic Realizations

Increasing SNR

θ

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Nonlinear Estimation and Ambiguity Functions

$$\hat{\theta} = \arg\max_{\theta} \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

$$\frac{1}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$$

Capon Ambiguity Function

θ_2 θ_3 θ_4 θ_1 θ_{M-1} θ_M

IE IE IE IE IE IE

NIE

θ

$$\frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

Stochastic Realizations

PDF of $\hat{\theta}$

Increasing SNR

θ

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Nonlinear Estimation and Ambiguity Functions

$$\hat{\theta}_k = \arg \; k\text{th largest peak} \, \underset{\theta}{} \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

$$\frac{1}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)}$$

Capon Ambiguity Function

NEW

θ_2 θ_3 θ_1 θ_M θ_{M+1}

IE

NIE   NIE   IE   IE   IE

θ

$$\frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

Stochastic Realizations

Increasing SNR

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Adaptation of Method of Interval Errors (MIE) MSE Prediction to Capon Algorithm

- **In general MSE can be written as the sum of two terms***

$$E\left\{\left(\hat{\theta}-\theta\right)^2\right\} = \Pr(\text{No Interval Error})E\left\{\left(\hat{\theta}-\theta\right)^2\middle|\text{No Interval Error}\right\}$$

$$+ \Pr(\text{Interval Error})E\left\{\left(\hat{\theta}-\theta\right)^2\middle|\text{Interval Error}\right\}$$

- **MSE for Deterministic (but unknown) Signal Parameters**

NEW ADAPTATION

$$\Theta = \{\text{True signal parameters are } \theta_k, k = 1,2,\text{K}, K\}$$

$$E\left\{\left(\hat{\theta}_k-\theta_k\right)^2\middle|\Theta\right\} \cong \left[1 - \sum_{m=K+1}^{K+M-1} p\left(\hat{\theta}_k = \theta_m\middle|\Theta\right)\right] \cdot \sigma_{\text{VB}}^2(\theta_k) + \sum_{m=K+1}^{K+M-1} p\left(\hat{\theta}_k = \theta_m\middle|\Theta\right) \cdot \left(\theta_m - \theta_k\right)^2$$

H. L. Van Trees 1968

Adaptation of Method of Interval Errors (MIE) MSE Prediction to Capon Algorithm

- **In general MSE can be written as the sum of two terms***

$$E\left\{(\hat{\theta} - \theta)^2\right\} = \Pr(\text{No Interval Error})E\left\{(\hat{\theta} - \theta)^2 \middle| \text{No Interval Error}\right\}$$

$$+ \Pr(\text{In} \qquad \qquad \qquad \text{or}\Big\}$$

- **MSE for Deterministic (but un**

NEW ADAPTATION

$$\Theta = \{\text{True signal parameters are } \theta_k, k = 1,2,\text{K }, K\}$$

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> **Error probabilities weight MSE transition from "No Information" region to "Asymptotic" region**

H. L. Van Trees 1968

Adaptation of Method of Interval Errors (MIE) MSE Prediction to Capon Algorithm

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$$E\left\{(\hat{\theta}-\theta)^2\right\} = \Pr(\text{No Interval Error})E\left\{(\hat{\theta}-\theta)^2 \middle| \text{No Interval Error}\right\}$$

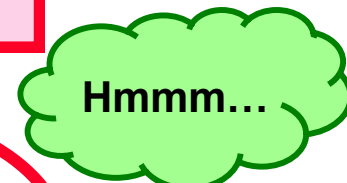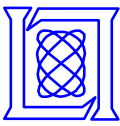$$+ \Pr(\text{Interval Error})E\left\{(\hat{\theta}-\theta)^2 \middle| \text{Interval Error}\right\}$$

Error probabilities weight MSE <u>transition</u> from "No Information" region to "Asymptotic" region

- **MSE for Deterministic (but unknown)**

NEW ADAPTATION

$$\Theta = \left\{\text{True signal parameters are } \theta_k, k = 1, 2, \text{K}, K\right\}$$

$$E\left\{(\hat{\theta}_k-\theta_k)^2 \middle| \Theta\right\} \cong \left[1 - \sum_{m=K+1}^{K+M-1} p(\hat{\theta}_k = \theta_m | \Theta)\right] \cdot \sigma_{\text{VB}}^2(\theta_k) + \sum_{m=K+1}^{K+M-1} p(\hat{\theta}_k = \theta_m | \Theta)(\theta_m - \theta_k)^2$$

Asymptotic MSE**

*H. L. Van Trees 1968
**Vaidyanathan, Buckley IEEE T-SP 1995
Stoica, Handel, Soderstrom SP 1995

Adaptation of Method of Interval Errors (MIE) MSE Prediction to Capon Algorithm

- **In general MSE can be written as the sum of two terms***

$$E\left\{(\hat{\theta}-\theta)^2\right\} = \Pr(\text{No Interval Error})E\left\{(\hat{\theta}-\theta)^2 \middle| \text{No Interval Error}\right\}$$

$$+ \Pr(\text{In}\ldots\text{or})$$

Error probabilities weight MSE _transition_ from "No Information" region to "Asymptotic" region

- **MSE for Deterministic (but un...**

NEW ADAPTATION

$$\Theta = \{\text{True signal parameters are } \theta_k, k = 1, 2, \text{K}, K\}$$

Hmmm…

$$E\left\{(\hat{\theta}_k - \theta_k)^2 \middle| \Theta\right\} \cong \left[1 - \sum_{m=K+1}^{K+M-1} p(\hat{\theta}_k = \theta_m | \Theta)\right] \cdot \sigma_{\text{VB}}^2(\theta_k) + \sum_{m=K+1}^{K+M-1} p(\hat{\theta}_k = \theta_m | \Theta)(\theta_m - \theta_k)^2$$

Asymptotic MSE**

***Difficult part* is obtaining error probabilities!**

*H. L. Van Trees 1968
**Vaidyanathan, Buckley IEEE T-SP 1995
Stoica, Handel, Soderstrom SP 1995

Applying Union Bound (UB) to Capon Algorithm Error Probabilities

- **Let true source angles be $\{\theta \,|\, \theta_1, \theta_2, \mathrm{K}, \theta_K\}$ and all other ambiguity function local maxima be denoted by set Φ.** **Defining** $\zeta_k \equiv \{\theta \,|\, \theta_k \cup \Phi\}$ ➡

- **The probability of interval error can be approximated by UB:**

$$p\left(\hat{\theta}_k = \theta_m \,\Big|\, \Theta\right) = 1 - p\left(\hat{\theta}_k \neq \theta_m \,\Big|\, \Theta\right) = 1 - \Pr\left\{ \bigcup_{\substack{n \in \zeta_k \\ n \neq m}} \left[P_{Capon}(\theta_n) > P_{Capon}(\theta_m) \,\Big|\, \Theta \right] \right\}$$

- **UB is the most widely used tool for calculation of error probabilities in Digital Communications**

  - **Approximation relies on <u>pairwise error probabilities</u>**

- **UB provides accurate predictions in the threshold region**

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UNION BOUND

Dominant Term

$$\Pr\left\{ \bigcup_{\substack{n \in \zeta_k \\ n \neq m}} \left[P_{Capon}(\theta_n) > P_{Capon}(\theta_m) \mid \Theta \right] \right\} \leq \sum_{\substack{n \in \zeta_k \\ n \neq m}} \Pr\left[P_{Capon}(\theta_n) > P_{Capon}(\theta_m) \mid \Theta \right] \approx \Pr\left[P_{Capon}(\theta_k) > P_{Capon}(\theta_m) \mid \Theta \right]$$

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Applying Union Bound (UB) to Capon Algorithm Error Probabilities

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UNION BOUND

Dominant Term

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Non-trivial Problem!

- **UB is the most widely used tool for calculation of error pro** in Digital Communications

$$p\left(\hat{\theta}_k = \theta_m \mid \Theta\right) = 1 - p\left(\hat{\theta}_k \neq \theta_m \mid \Theta\right) \approx \Pr\left[P_{Capon}(\theta_m) > P_{Capon}(\theta_k) \mid \Theta\right]$$

- **UB provides accurate predictions in the threshold region**

Capon-MVDR Algorithm
Pairwise Error Probabilities

NEW *

$$\Pr\left[P_{Capon}(\theta_m) > P_{Capon}(\theta_k) \,\middle|\, \Theta\right] =$$

$$0.5 \cdot \left\{1 + \text{sign}\left[\lambda_{\Delta,1}(-1)\right]\right\} - \text{sign}\left[\lambda_{\Delta,1}(-1)\right] \cdot \Omega\left[-\left(\frac{\lambda_{\Delta,2}(-1)}{\lambda_{\Delta,1}(-1)}\right), L - N + 2\right]$$

*Exact pairwise error probability given by
simple finite sum involving no numerical integration!*

- **Assume data is complex Gaussian and define the following matrices**

$$\mathbf{V} = \left[\mathbf{v}(\theta_m), \mathbf{v}(\theta_k)\right] \quad \mathbf{Q}\mathbf{R}^{-1/2}\mathbf{V} = \begin{bmatrix} \Delta_{2\times2} \\ \mathbf{0} \end{bmatrix} \text{ and function } \quad \Omega(x, M) \equiv \frac{x^M}{(1+x)^{2M-1}} \sum_{k=0}^{M-1} \binom{2M-1}{k+M} x^k.$$

- **The quantities necessary for error probability calculation are given by:**

$$\Delta \equiv \left[\mathbf{a} \mid \mathbf{b}\right] \implies \mathbf{b}\mathbf{b}^H + F \cdot \mathbf{a}\mathbf{a}^H \equiv \mathbf{Q}_\Delta(F) \begin{bmatrix} \lambda_{\Delta,1}(F) & 0 \\ 0 & \lambda_{\Delta,2}(F) \end{bmatrix} \mathbf{Q}_\Delta^H(F)$$

$$P_{Capon}(\theta) = \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$
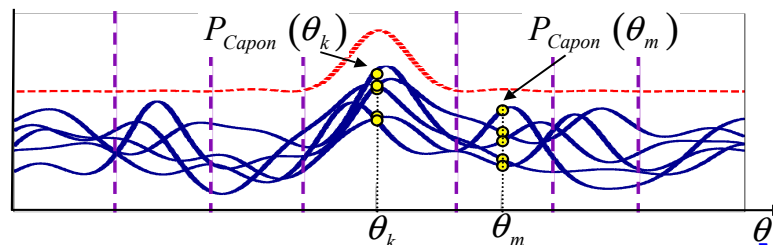
$P_{Capon}(\theta_k)$ $P_{Capon}(\theta_m)$

*Richmond, Paper to appear IEEE T-SP, ICASSP '04

Outline

- **Introduction**
- **Capon Algorithm**
- **Theory of MSE Prediction**
✓ - **Numerical Examples**
  - **Threshold SNR Predictions**
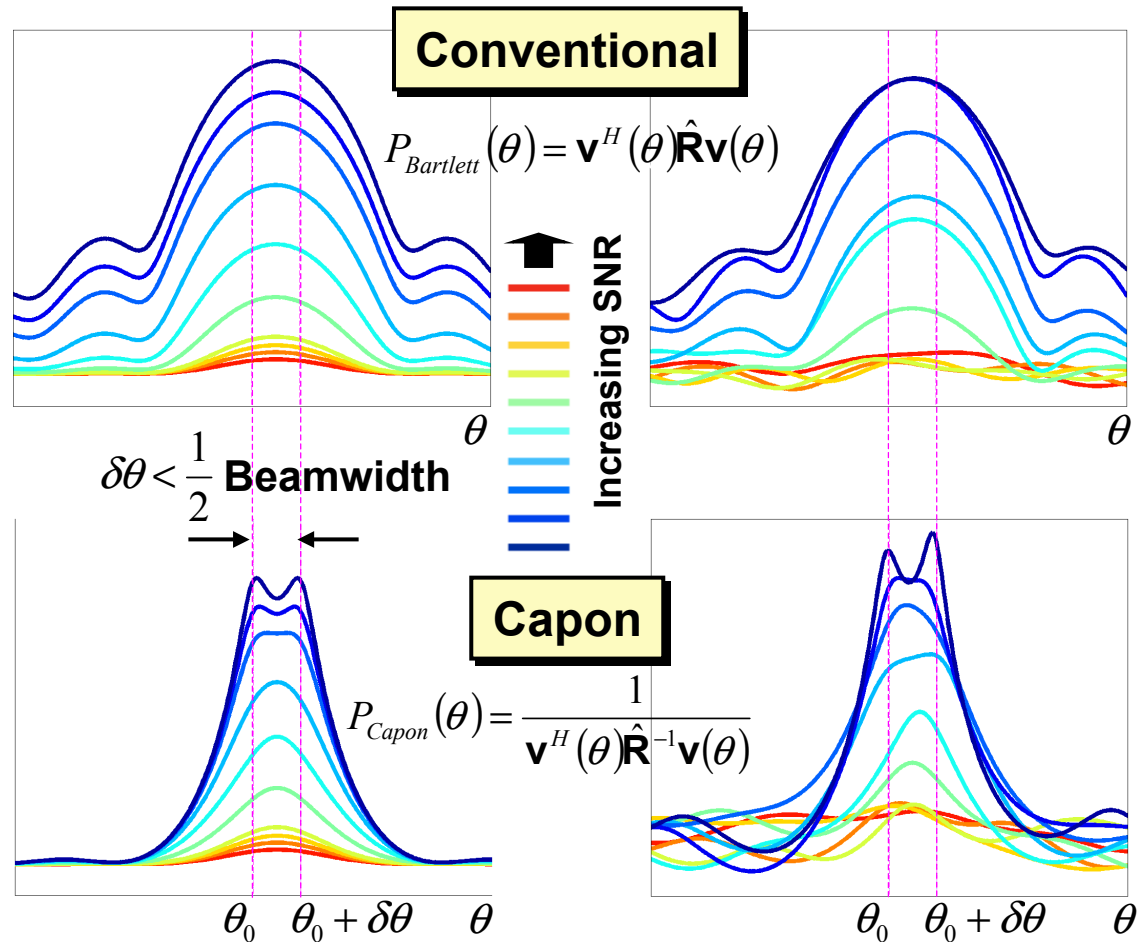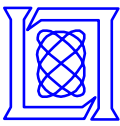 - **Probability of Resolution**
- **Conclusions**

Single Signal Broadside to Array in Spatially White Noise
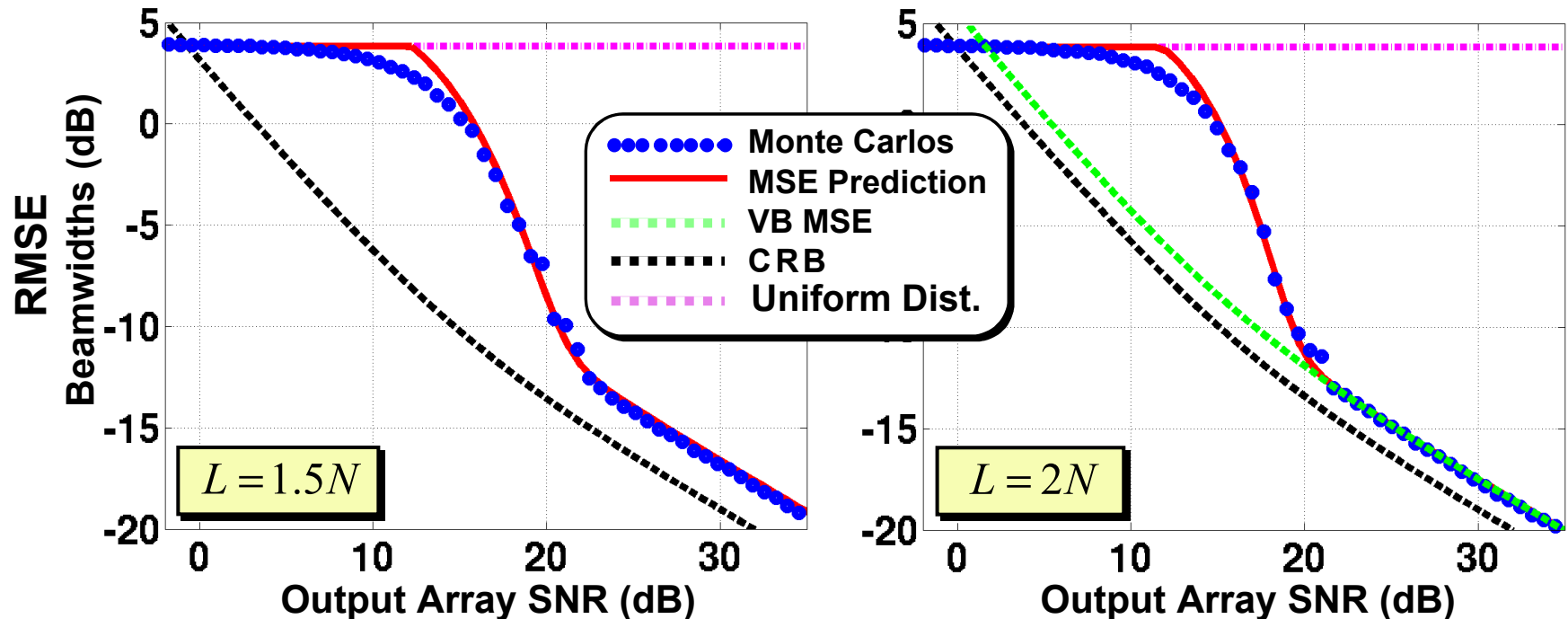
- *N*=18 **element uniform linear array (ULA),** $(\lambda/2.25)$ **element spacing**
  - **3dB Beamwidth ≈ 7.2 degs**
  - **0dB white noise, True Signal @ 90 degs (broadside)**
  - **Search space** $\theta \in [60°,120°]$, **8000 Monte Carlo simulations**
- **MIE provides accurate MSE prediction well into threshold region**

Single Signal Broadside to Array in Spatially White Noise
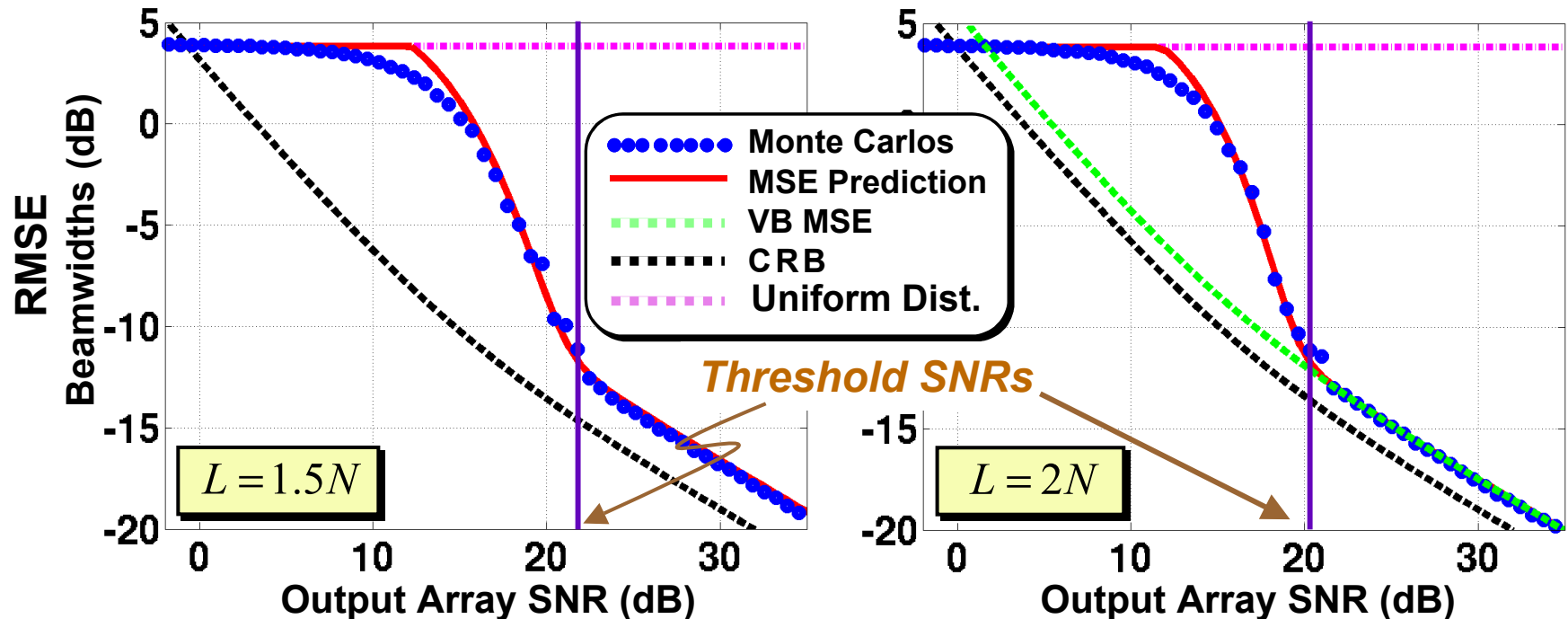
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Two Well Separated Signals in Spatially White Noise: 90° (broadside) and 70°
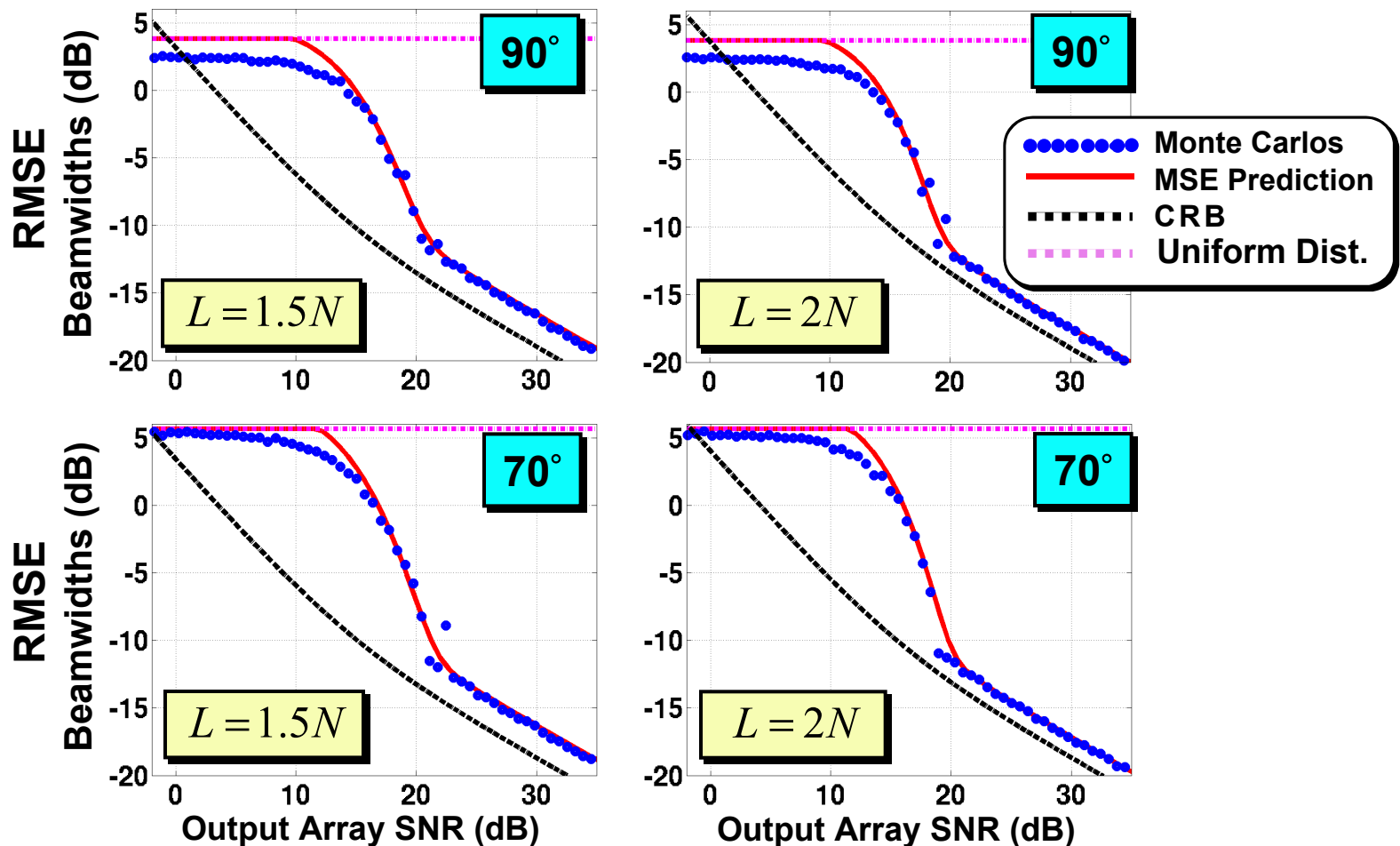
- Now add equal power source @ 70° to same scenario and array
- MIE accurately predicts threshold region MSE performance of multiple sources

Two Well Separated Signals in Spatially White Noise: 90° (broadside) and 70°
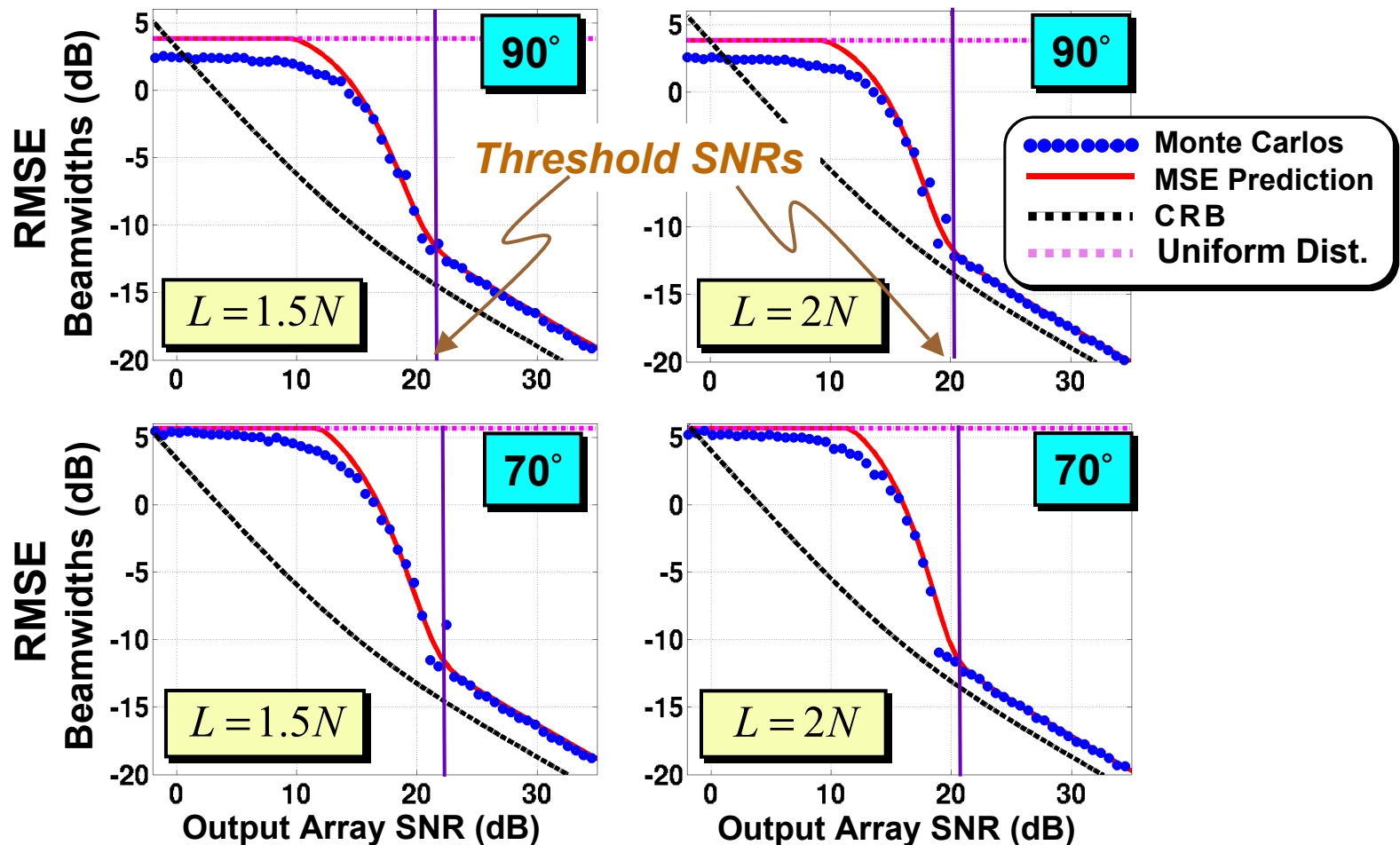
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MRLA ● ●   ●   ●

ULA ● ● ● ● ● ● ●

Beampattern

Tilted MRLA

Tilt Angle
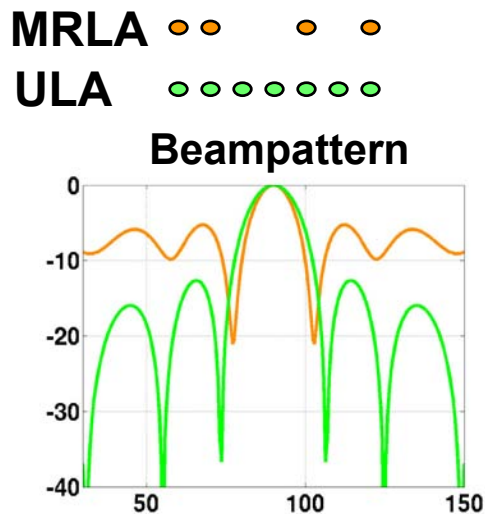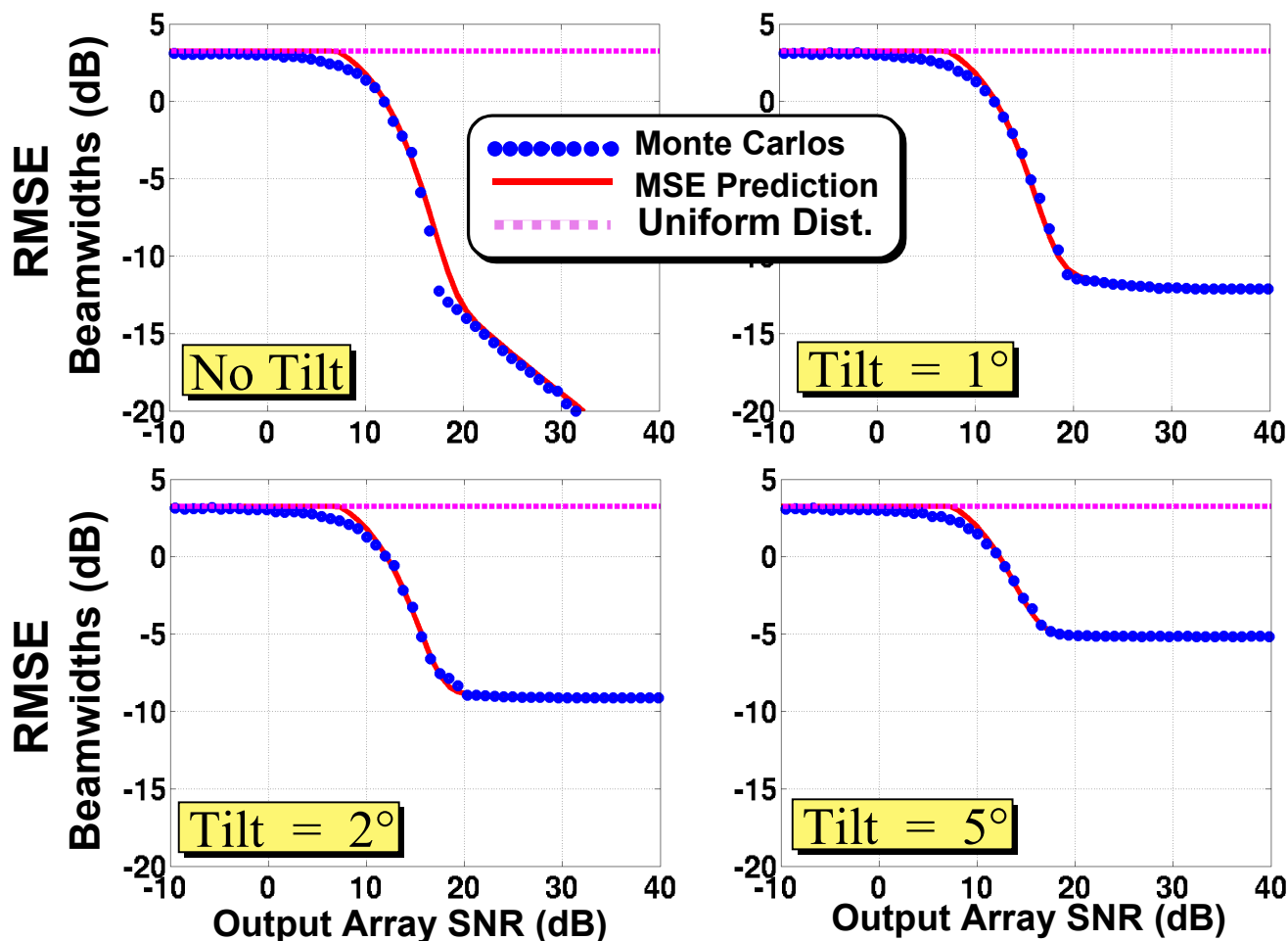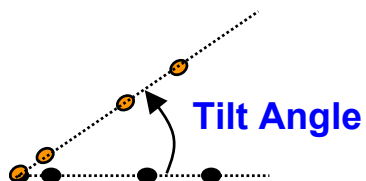
- **Angle estimation of broadside source: $L = 12$ spatial snapsots**
- **Array tilt introduces mismatch resulting in asymptotic bias**

Outline

- **Introduction**
- **Theory of MSE Prediction**
- **Numerical Examples**
  - Threshold SNR Predictions
  - ✓ **Probability of Resolution**
- **Conclusions**

Closely Spaced Sources and Resolution

- **Two closely spaced signals are resolved if estimated spectrum yields two distinct peaks at the location of both signals**

- **Resolution of Conventional / FFT spectrum dictated by aperture length of array (window length) independent of SNR**

- **Capon algorithm has resolution capability superior to Conventional approach**

    - **Influenced by aperture length and degrees of freedom**

    - **Improves with SNR**

Ambiguity Function **Stochastic Realizations**

Conventional

$$P_{Bartlett}(\theta) = \mathbf{v}^H(\theta)\hat{\mathbf{R}}\mathbf{v}(\theta)$$

Increasing SNR

$$\delta\theta < \frac{1}{2}\,\textbf{Beamwidth}$$

Capon

$$P_{Capon}(\theta) = \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$

$\theta_0 \quad \theta_0 + \delta\theta \quad \theta$

Capon Algorithm Probability of Resolution: A Two Point Measure

Capon Stochastic Realizations

$$P_{Capon}(\theta) = \frac{1}{\mathbf{v}^H(\theta)\hat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}$$
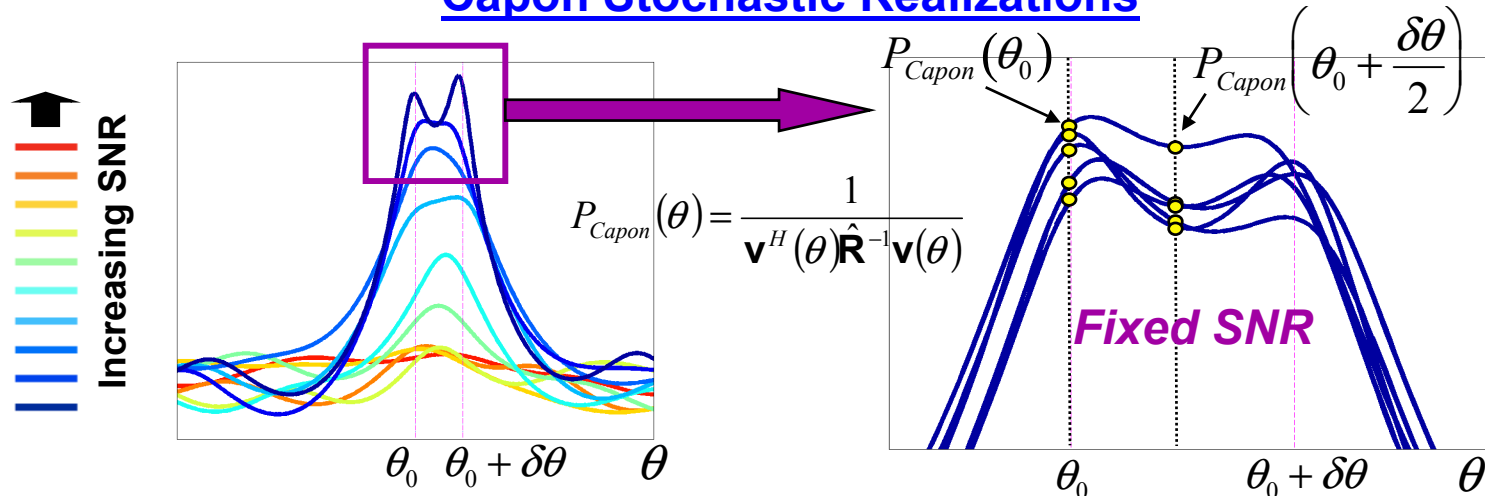
- A two point measure of the Capon probability of resolution can be defined by a modified pairwise error probability calculation.

- Let desired "dip" parameter ρ be defined such that $0 \le \rho \le 1$, (e.g., $\rho = 0.5$ represents a 3dB dip). It can be shown that*

NEW *

$$\Pr\left[P_{Capon}\left(\theta_0 + \frac{\delta\theta}{2}\right) \le \rho \cdot P_{Capon}(\theta_0) \,\middle|\, \Theta \right] =$$

$$0.5 \cdot \left\{1 + \text{sign}\left[\lambda_{\Delta,1}(-1/\rho)\right]\right\} - \text{sign}\left[\lambda_{\Delta,1}(-1/\rho)\right] \cdot \Omega\left\{-\left[\frac{\lambda_{\Delta,2}(-1/\rho)}{\lambda_{\Delta,1}(-1/\rho)}\right], L - N + 2\right\}$$

Richmond, Paper to appear IEEE T-SP, ICASSP '04

Capon Probability of Resolution:
Array Position Uncertainty (Mismatch)
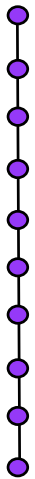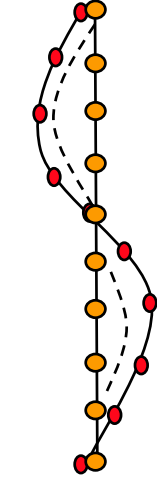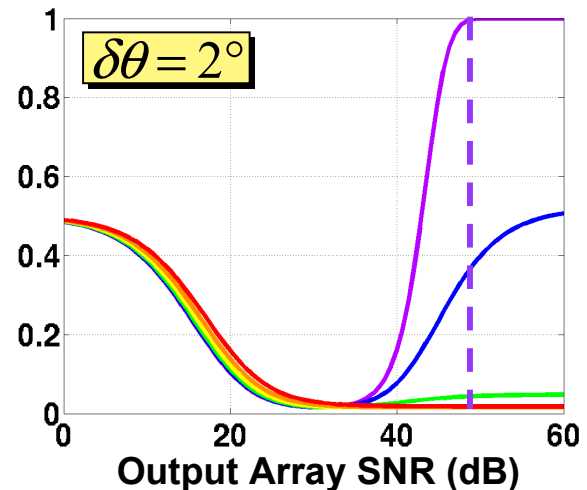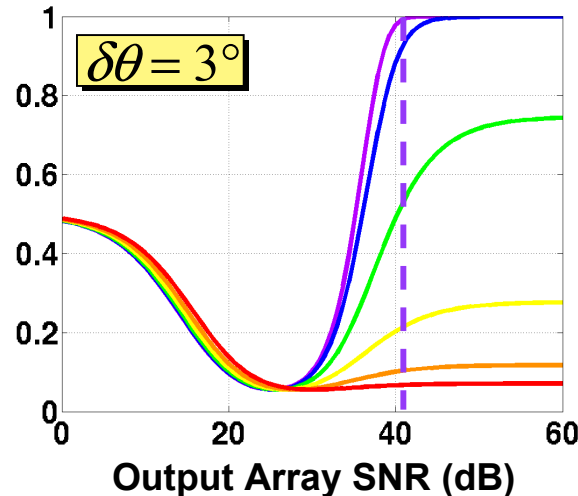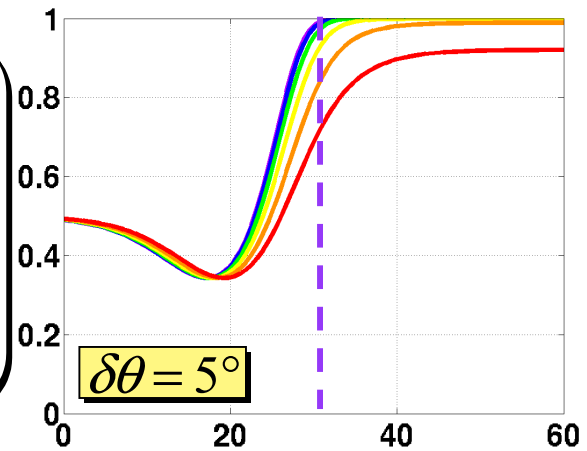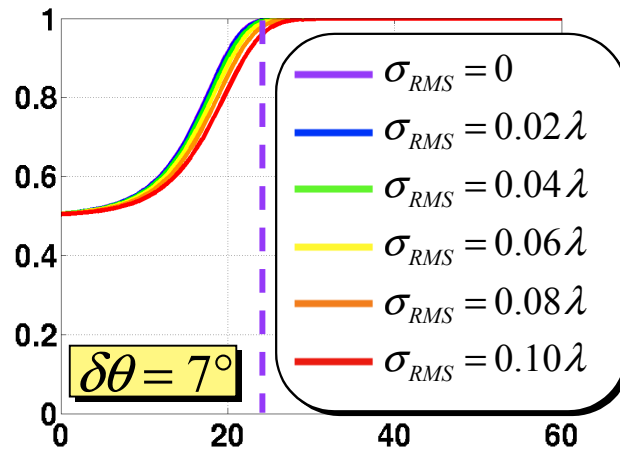
- N=18 element ULA, ($\lambda/2.25$) element spacing, 3dB Beamwidth ≈ 7.2 degs, 0dB white noise
- First signal @ 90 degs (broadside) and second signal @ 90+$\delta\theta$ degs; L=2N;

Outline

- **Introduction**
- **Capon Algorithm**
- **Theory of MSE Prediction**
- **Numerical Examples**
- **Conclusions**

Conclusions

- **This new theory provides powerful tools for the design and analysis of any system employing the Capon algorithm**

    – **Threshold region MSE predictions account for multiple well separated sources, finite sample effects, colored data covariance, and signal mismatch**

    – **Pairwise error probabilities given by simple finite sums involving no numerical integration**

- **Results rival the best of Bayesian bounds (Ziv-Zakai, Weiss-Weinstein, etc.) as means of predicting threshold region performance**

    – **Signal parameters remain deterministic**

    – **Ease of accounting for nuisance parameters (unknown data covariance)**

    – **Fast and computationally efficient**

    – **Accuracy of predictions / Algorithm specific**

- **A new exact two point measure of the Capon algorithm probability of resolution introduced accounting for finite training, arbitrary data covariance and signal model mismatch**

    – **Accurate prediction of the SNR necessary for closely spaced signals to be resolved**

Where to now ?

Current

- **Capon signal model mismatch analysis applied to adaptive matched field processing with N. Lee**
    - **ASAP Poster / To be submitted to JASA 2004**
- **Joint probability density function (PDF) of Conventional Bartlett and Capon spectral estimators**
    - **To be submitted to IEEE T-SP 2004**
- **Companion analysis of maximum-likelihood signal parameter estimation with estimated colored noise covariance**
    - **Submitted to IEEE T-IT August 2003**
- **PDF Diagonally loaded Capon with M. Chiani and M. Win**
- **Applications to localization of brain activity with B. D. Van Veen**

Future

- **Adapt MSE predictions to handle closely spaced sources scenario**
- **Exact PDF of position of Capon algorithm global max**
- **Implications to MIMO frequency offset and channel estimation**
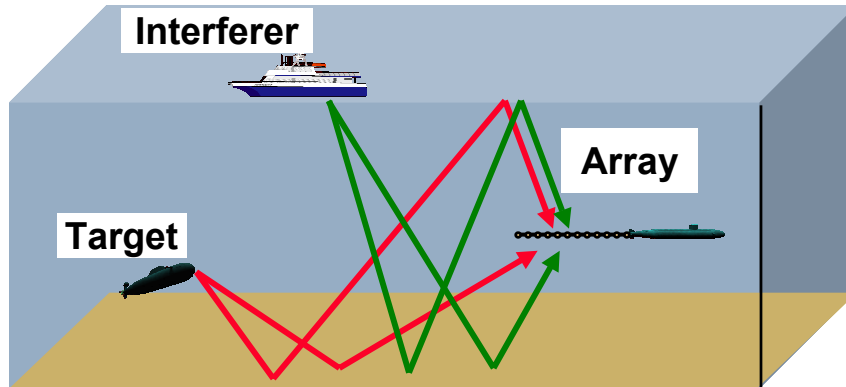- **Tracking**

Thank You!

Backups

Problem Statement

**Shallow-Water Multipath Propagation:**


Matched field processing (MFP)
models acoustic multipath propagation
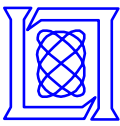to enable 3-D source localization

- **Goal**: quantify what SNR's are required for acceptable MFP localization
- **Method**: MFP mean-squared error prediction that accounts for
  - Ambiguities in MFP output
  - Finite sample adaptive training
  - Mismatch
  - Colored noise (discrete interferers)

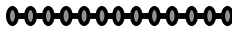**"Populated" Array MFP Beampattern
(Vertical Array)**
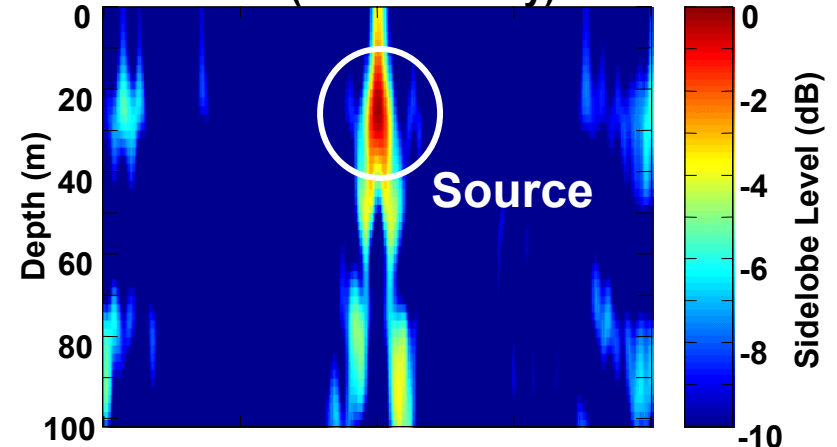
**"Sparse" Array MFP Beampattern
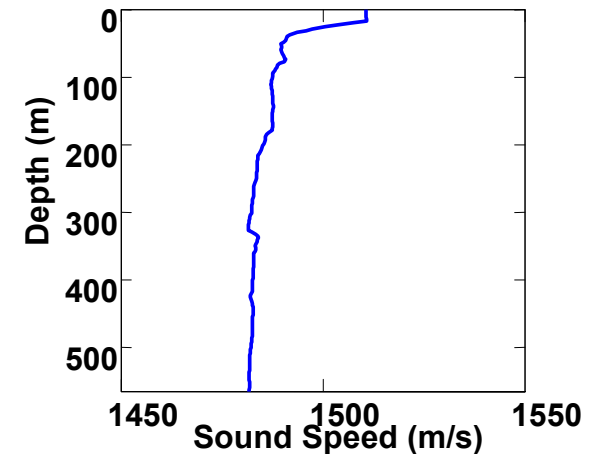(Horizontal Array, Endfire)**
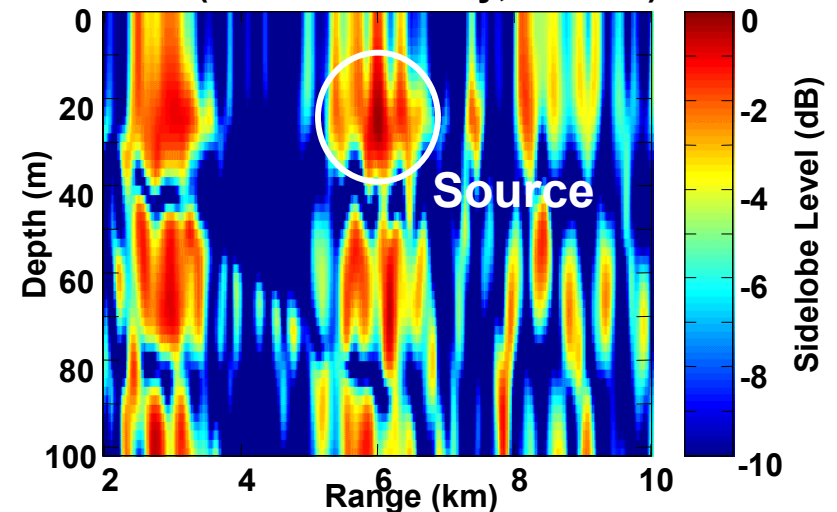
Simulation Parameters:
Signal in White Noise

- **Uniform (horizontal) line array, N = 41, element spacing   dx = 15 m, total length 600 m, depth 100 m**

  oooooooooooooo

- **Southern California environment (water depth 568 m)**
- **Processing frequency 50 Hz**
- **Signal: 6 km range, 25 m depth, 0° bearing (endfire)**
  - **Phone level power σ_s^2 = SL - TL (dB) varies**
- **Noise: NL = σ_n^2 = 70 dB, complex white Gaussian noise**
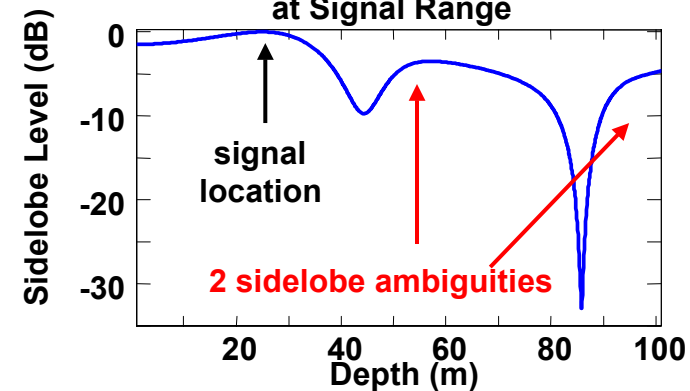- **Search space: 2-10 km range, 1-101 m depth, 0° bearing**

Sound Speed Profile

Range Beampattern at Signal Depth

signal location

18 sidelobe ambiguities
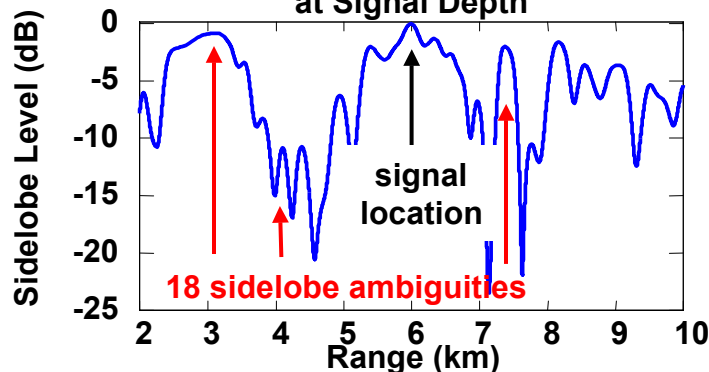
Depth Beampattern at Signal Range

signal location
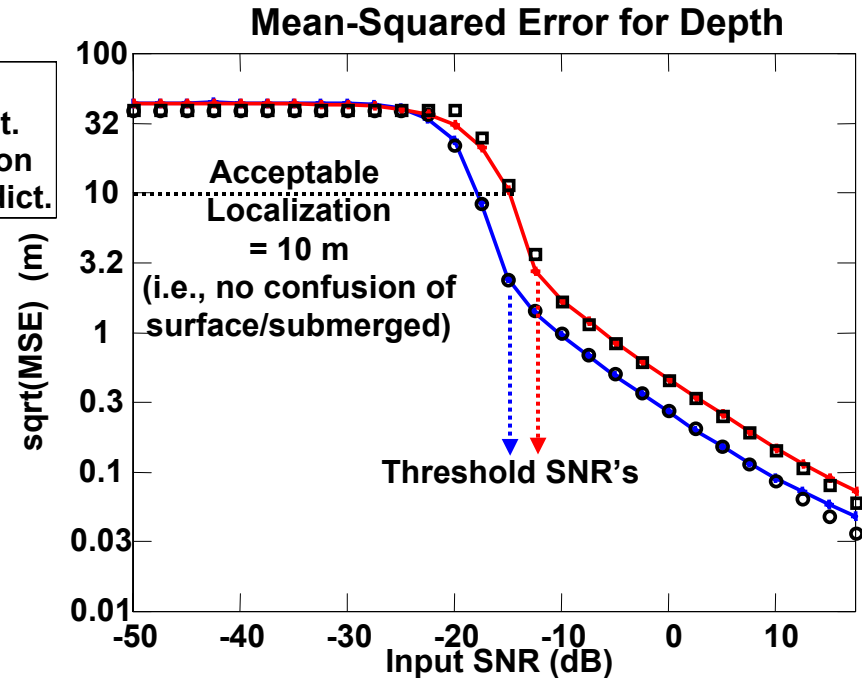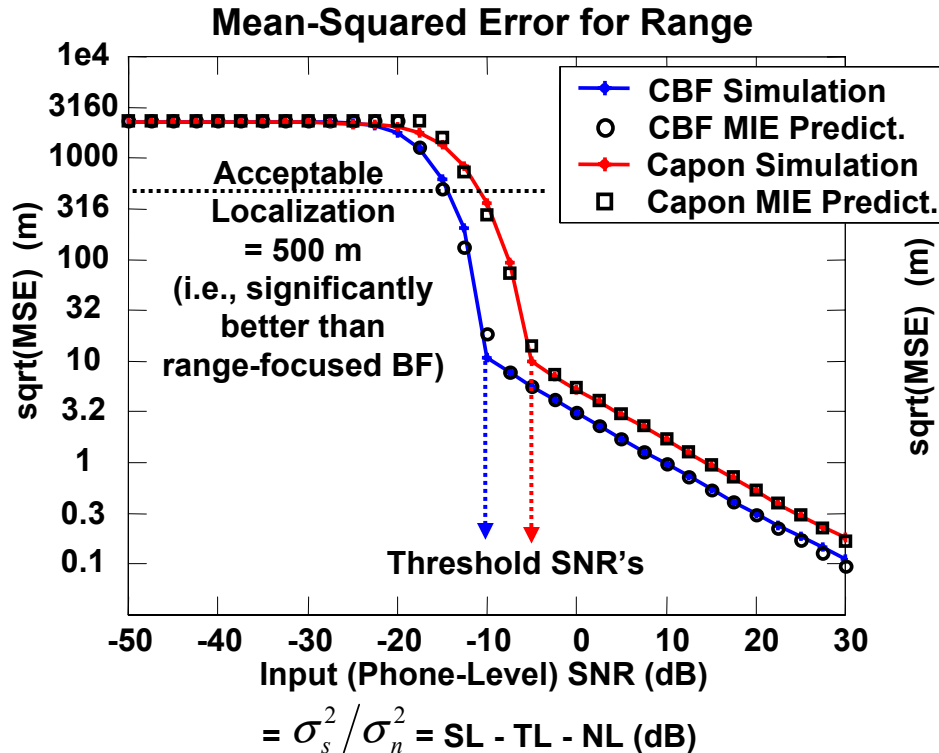
2 sidelobe ambiguities

Assume 1-D parameter estimation here (simple extension to 2-D)

Simulation Results:
Signal in White Noise

L = 60 snapshots, N = 41 elements
4000 Monte Carlo runs for each SNR

For white noise, MLE and CBF give identical estimates!

Mean-Squared Error for Range

Acceptable Localization = 500 m (i.e., significantly better than range-focused BF)

- CBF Simulation
- CBF MIE Predict.
- Capon Simulation
- Capon MIE Predict.

Threshold SNR's

Input (Phone-Level) SNR (dB)

$$= \sigma_s^2 / \sigma_n^2 = \text{SL - TL - NL (dB)}$$

Mean-Squared Error for Depth

Acceptable Localization = 10 m (i.e., no confusion of surface/submerged)

Threshold SNR's

Input SNR (dB)

- **MIE predictions accurate within 1-2 dB for entire SNR range**
- **For white noise, CBF outperforms Capon (training effects)**
- **Limits on acceptable performance occur in threshold region**